Critical States in Disordered Superconducting Films

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Abstract. When subject to a pair-breaking perturbation, the pairing susceptibility of a disordered superconductor exhibits substantial long-ranged mesoscopic fluctuations. Focusing on a thin film subject to a parallel magnetic field, it is proposed that the quantum phase transition to the bulk superconducting condensate may be preempted by the formation of a glass-like phase with multi-fractal correlations of a complex order parameter. Although not universal, we argue that such behavior may be a common feature of quantum critical phenomena in disordered environments.

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Several years ago, in an inspiring sequence of papers, it was proposed by Spivak and Zhou [1, 2] that mesoscopic fluctuations can significantly influence the nature of the transition to superconductivity in the vicinity of the upper critical field, H_{c2}. Considering a magnetic field applied perpendicular to a film, it was argued that the transition to the mixed phase is mediated by the formation of a superconducting 'glass-like' phase realized through the random Josephson coupling of droplets or domains nucleated at fields in excess of H_{c2} [1]. Subsequently, focusing this time on the properties of a superconducting thin-film subject to a parallel field, qualitatively similar conclusions were drawn [2]. Recently, Galitski and Larkin [3] extended these ideas exploring the interplay of the proximity effect and quantum phase fluctuations on the integrity of the superconducting glass in the perpendicular field geometry. Here, focusing on a weakly disordered thin-film subject to a parallel magnetic field H, we will offer a different perspective on the character of the superconducting phase close to the quantum critical point. In this geometry, taking the film thickness d to be smaller than the penetration depth, the field lines enter the sample and effect a mechanism of pair-breaking leading to the gradual suppression of the superconducting order parameter. However, in contrast to Ref. [2], we will find it convenient to limit considerations to the range $d \gg g_L \lambda_F \xi_0/\ell$, where g_L denotes the Landé g-factor, λ_F is the Fermi wavelength, ℓ the elastic mean free path, and $\xi_0 = \sqrt{D/\Delta_0}$ (with the classical diffusion constant $D = v_F\ell/2$) represents the diffusive superconducting coherence length of the unperturbed system. In this limit, where the impact of Zeeman splitting can be safely neglected, the transition to the (gapless) superconducting phase is second order and described by a mean-field theory of Abrikosov-Gor'kov (AG) type [5].

Starting with a microscopic BCS Hamiltonian for the disordered, symmetry-broken system, one can present the quantum partition function for an *individual realization* of the impurity potential as a functional field integral $\mathcal{Z} = \int D[\Delta,\bar{\Delta}]\ e^{-S_\Delta}$, where, to quadratic order in Δ,

$$S_\Delta = \sum_{\omega_m} \int d\mathbf{r}\, d\mathbf{r}'\ \bar{\Delta}_{\omega_m}(\mathbf{r}) \left[\frac{1}{\lambda_{\rm BCS}}\delta(\mathbf{r}-\mathbf{r}') - \Pi_{\omega_m}(\mathbf{r},\mathbf{r}')\right] \Delta_{\omega_m}(\mathbf{r}')\ . \tag{1}$$

Here, $\lambda_{\rm BCS}$ denotes the coupling constant of the BCS interaction while $\Pi_{\omega_m}(\mathbf{r},\mathbf{r}') = T\sum_{\epsilon_n} G_{\epsilon_n}(\mathbf{r},\mathbf{r}')G_{\omega_m-\epsilon_n}(\mathbf{r},\mathbf{r}')$, where $G_{\epsilon_n}(\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|(i\epsilon_n - \hat{H})^{-1}|\mathbf{r}'\rangle$ is the sample specific Matsubara Green function, represents matrix elements of the (generically complex) Hermitian pairing susceptibility. Since we are primarily interested in the character of the transition, and not properties deep in the superconducting phase, the influence of the non-linear terms in the Hamiltonian can be neglected.

Leaving aside the influence of dynamical fluctuations of the order parameter, in the saddle-point approximation, the transition to the superconducting phase is signaled by the appearance of solutions of the linear equation

$$\int d\mathbf{r}'\,\Pi_0(\mathbf{r},\mathbf{r}')\chi(\mathbf{r}') = \nu\gamma\,\chi(\mathbf{r})\ , \tag{2}$$

with eigenvalues $\gamma \geq (\lambda_{\rm BCS}\,\nu)^{-1}$, defining ν as the density of states (DoS) per spin of the normal phase (assumed constant). In the leading approximation, the influence of disorder on the symmetry-broken system can be explored by replacing the pairing susceptibility by its impurity average, $\langle\hat{\Pi}\rangle_V$. In the specified thin-film geometry, the 'paramagnetic term' due to the parallel magnetic field is negligible, and $\langle\hat{\Pi}\rangle_V$ takes the form

$$\langle \Pi_0(\mathbf{r},\mathbf{r}')\rangle_V = 4\pi\nu T \sum_{\epsilon_n} \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{D\mathbf{q}^2 + 2|\epsilon_n| + 2/\tau_H}\ , \tag{3}$$

where $1/\tau_H = De^2(Hd)^2/6$ denotes the pair-breaking rate associated with the orbital motion of the particle in the external field [4]. In this approximation, when substituted into (2), one finds that the system becomes unstable against the formation of a homogeneous condensate when $1/\tau_H$ fulfills the AG condition [5], $\ln(T_c/T_c^0) = \psi(1/2) - \psi(1/2 + 1/2\pi\tau_H T_c)$, where T_c^0 is the transition temperature of the unperturbed system, and $\psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the digamma function. In particular, when the pair-breaking rate equals to the order parameter of the unperturbed system, $2/\bar{\tau}_H^c = \Delta_0 \equiv 2\times 0.88\,T_c^0$, the bulk superconducting phase is destroyed altogether.

However, as emphasized by Spivak and Zhou [2], mesoscopic fluctuations of the pairing susceptibility can significantly influence the character of the transition. In particular, at the transition, the condensate wave function may acquire a texture in both *phase* as well as amplitude: Focusing on the static component, the complex pairing susceptibility exhibits spatial mesoscopic fluctuations $\delta\Pi_0(\mathbf{r},\mathbf{r}') = \Pi_0(\mathbf{r},\mathbf{r}') - \langle\Pi_0(\mathbf{r},\mathbf{r}')\rangle_V$. For a given configuration of the quenched impurity potential, a system may lower its energy by forming a condensate where the corresponding wave function is random and moreover, because of the presence of the magnetic field, complex. In this case, an arrangement of supercurrent loops can optimally screen the applied field. In particular, the condensate can exploit the quenched spatial fluctuations of the pairing susceptibility to initiate a transition to a superconducting phase ahead of that predicted by the mean-field estimate above. However, in the bulk system, such an arrangement of supercurrent loops can be tolerated only in the near vicinity of the transition. Away from the quantum critical point, the energy cost in suppressing the amplitude of the order parameter in the superconducting phase becomes prohibitively high and a condensate of uniform phase must develop.

To explore the influence of mesoscopic fluctuations on the nature of the transition one can proceed along complimentary paths: Firstly, one can develop a scheme perturbative in fluctuations $\delta\Pi$, treating typical field configurations of the order parameter $\Delta_0(\mathbf{r})$ associated with the *impurity averaged susceptibility* as a variational *Ansatz*. When the scattering rate $1/\tau_H$ is in excess of $1/\bar{\tau}_H^c$, the superconducting order parameter $\Delta_0(\mathbf{r})$ exhibits spatial fluctuations around a zero mean with a correlation length $L_D = \xi_0(\bar{\tau}_H^c/\tau_H - 1)^{-1/2}$ diverging at the critical point. Regarding these spatial configurations as domains or "droplets" of ordered phase with a characteristic length scale L_D, fluctuations $\delta\hat{\Pi}_0$ impose a long-ranged, complex, random Josephson coupling — a superconducting "gauge glass" [2]. While such a variational approach is naively applicable close to the bulk transition — see below — one might expect it to underestimate the impact of optimal fluctuations of the random susceptibility.

Alternatively, treating the optimal fluctuations more accurately, one can pursue a more direct 'mean-field' approach, seeking explicit solutions of the stochastic saddle-point equation (2) (cf. Ref. [3]). In this case, guided by the intuition afforded by the properties of band tail states of random Hamiltonian operators, one might expect the behavior close to the critical point to be dominated by spatially localized field configurations of $\Delta_0(\mathbf{r})$ associated with rare or *optimal fluctuations* of the random operator $\hat{\Pi}_0$. Ignoring the potential impact of spatial and dynamical fluctuations of the order parameter around these saddle-point configurations, such an approach would suggest that the formation of the homogeneous superconducting condensate phase is preempted by the nucleation of domains of ordered phase, the implication being that the bulk transition will be mediated by phase ordering of Josephson coupled islands. Crucially, since the droplets reflect fluctuations of the complex pairing susceptibility, the condensate wave function would itself be complex — a superconducting 'glass-like' phase of the type described by Ref. [2].

The distinction between the two approaches may to some extent be semantic. While the former approach may underestimate the capacity of the system to form 'droplets' of condensate, the nature of the bulk transition may rely more sensitively on the properties of the gauge glass: the geometry and rigidity of the superconducting domains is less important than the random Josephson coupling between the domains arising from the fluctuations. However, in the parallel field geometry, when taking into account the long-ranged nature of the correlations of $\hat{\Pi}_0$, we will argue that the transition affords a potentially different interpretation. Specifically, we will show that fluctuations $\delta\Pi_0(\mathbf{r},\mathbf{r}')$ engage *long-ranged* diffusion modes and, as such, are correlated over long distances. Such long-range correlations discriminate the spectral properties of the pairing susceptibility from those of usual short-ranged random impurity models. In particular, in contrast to band tail states in low-dimensional Hamiltonian systems, the transition between the tail state region and the 'extended' bulk state region is not mediated by a region of weakly localised states but rather it is abrupt: Even states close to the band edge of $\hat{\Pi}_0$ are power-law extended. If the condensate acquires the texture of these states, they will in turn inherit 'critical' spatial correlations.

To explore in detail the spectral properties of the (static) pairing susceptibility $\hat{\Pi}_0$, one requires information about the full distribution function of matrix elements, $P[\Pi_0]$. Clearly, being of two-particle character, such a programme seems unfeasible — at least in the two-dimensional geometry considered here. Instead, we will choose to characterize the scale of fluctuations through the variance of Π_0 which can be estimated diagrammatically. Here, to be consistent, one must discard fluctuations $\delta\Pi_0(\mathbf{r},\mathbf{r}')$ occurring on a length scale $|\mathbf{r}-\mathbf{r}'|$ shorter than the magnetic diffusion length $L_H = \sqrt{D\tau_H}$. Such fluctuations, which effect equally matrix elements of the unperturbed superconducting system, demand a more careful consideration of the influence of disorder on the pairing interaction itself. These effects are irrelevant to, and reach well-beyond, the present discussion. Although the ensemble average has all but vanished, on length scales $|\mathbf{r}-\mathbf{r}'| \gg L_H$, one finds that the matrix elements of the pairing susceptibility become complex and exhibit significant *long-ranged fluctuations*. Taking into account the leading contribution from the impurity diagrams depicted in Fig. 1a, a diagrammatic estimate of their magnitude obtains $\langle\delta\Pi_0(\mathbf{r}_1,\mathbf{r}_2)\delta\Pi_0(\mathbf{r}_3,\mathbf{r}_4)\rangle_V \simeq \tilde{\delta}(\mathbf{r}_1-\mathbf{r}_4)\tilde{\delta}(\mathbf{r}_2-\mathbf{r}_3)C(\mathbf{r}_1-\mathbf{r}_2)$, where

$$C(\mathbf{r}-\mathbf{r}') \sim \frac{\nu^2}{g^2}\left(\frac{\xi_0}{|\mathbf{r}-\mathbf{r}'|}\right)^4 , \tag{4}$$

with $g = \nu D$ the dimensionless conductance of the normal two-dimensional system. Indeed, a similar result was obtained by Spivak and Zhou [1] for the perpendicular field geometry: the difference between the parallel and perpendicular field orientations is manifest only in the short-ranged Cooperon contributions which dress the vertices, giving rise to the envelope functions of width $\sim L_H$ that we denoted by $\tilde{\delta}(\mathbf{r})$ — the long-range scaling is inherited from the field-insensitive diffuson content. While, for coordinates $\mathbf{r}_i$ separated by more than L_H, matrix elements $\delta\Pi_0(\mathbf{r}_1,\mathbf{r}_2)$ remain statistically uncorrelated from those of $\delta\Pi_0(\mathbf{r}_3,\mathbf{r}_4)$, it may be confirmed that the correlation function $\langle\delta\Pi_0(\mathbf{r},\mathbf{r})\delta\Pi_0(\mathbf{r}',\mathbf{r}')\rangle_V$ (Fig. 1b) also scales as $C(\mathbf{r}-\mathbf{r}')$.

Now, in the general two-dimensional system, the eigenfunctions of a typical random Hermitian operator exhibit a continuous evolution from strong localisation below the band edge of the non-disordered system, to weak localisation above. Applied to the pairing susceptibility, such behaviour would motivate a theory of the transition based on phase ordering of domains large enough to survive the effects of quantum fluctuations.

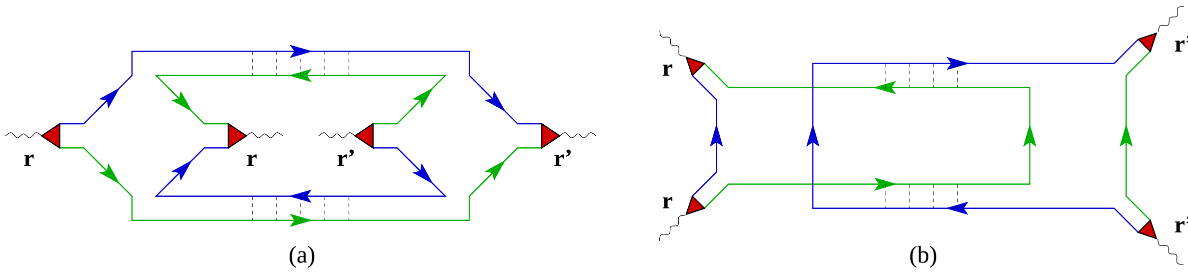

Figure 1. Impurity diagrams providing the leading contribution to the variance of the matrix element $\delta\Pi_0(\mathbf{r},\mathbf{r}')$. In both cases, the long-ranged character of the correlations can be ascribed to 'diffuson' contributions which are insensitive to magnetic field. The influence of the magnetic field is imposed through the short-ranged 'Cooperon' ladders which dress the vertices.

However, these considerations overlook the potential significance of the long-ranged power-law correlations of $\delta\Pi_0(\mathbf{r},\mathbf{r}')$. In the present case, the complex random matrix elements have a strength which decays only as a power-law, with an exponent β that coincides with the dimensionality D, i.e. $\beta = D = 2$ in the thin-film geometry. Previous investigations by Levitov [6, 7] have revealed that such a dependence places the system in a 'critical regime' where the bulk eigenstates of the susceptibility are neither fully extended nor exponentially localized. Rather, such states $\chi_\alpha(\mathbf{r})$ exhibit a multi-fractal structure all the way down to the 'band edge' with moments $\langle|\chi_\alpha(\mathbf{r})|^{2q}\rangle_\Pi \sim L^{-2q+\eta_q}$ characterized by a set of exponents η_q, and power-law spatial correlations,

$$f(\mathbf{r}-\mathbf{r}') \equiv L^4\langle|\chi_\alpha^2(\mathbf{r})\chi_\alpha^2(\mathbf{r}')|\rangle_\Pi \sim \left(\frac{L}{|\mathbf{r}-\mathbf{r}'|}\right)^{\eta_2} . \tag{5}$$

At the level of the saddle-point, the transition to the *bulk* superconducting phase may not, after all, be a problem of optimal fluctuations.

To make the analysis quantitative, one can use the variance (4) to characterize the distribution of fluctuations. On scales $|\mathbf{r}-\mathbf{r}'| \gg L_H$, let us suppose that matrix elements are specified by a Gaussian distribution, $P[\Pi_0] = \exp[-\frac{1}{2}\int d\mathbf{r}\, d\mathbf{r}'\ |\delta\Pi_0(\mathbf{r},\mathbf{r}')|^2/C(\mathbf{r}-\mathbf{r}')]$, where we include only the first type of correlations discussed above (see Fig. 1(a)). On length scales smaller than L_H, we will assume to be cautious that the matrix elements simply coincide with their impurity average. (Indeed, such an assumption underestimates the renormalisation of the quantum critical point.) Although there is no reason to expect the distribution to be Gaussian, an estimate of the leading contribution to the higher cumulants is compatible with the *Ansatz* for $P[\Pi_0]$. With this distribution, the rearrangement of the DoS implied by the long-ranged fluctuations of $\hat{\Pi}_0$ can be estimated within the self-consistent Born approximation (SCBA). Defining the impurity averaged Green function, $\hat{\mathcal{G}}_0 = \langle(\gamma_+ - \hat{\Pi}/\nu)^{-1}\rangle_\Pi$, where $\gamma_+ \equiv \gamma + i0$, the SCBA translates to the condition

$$-\frac{g_0}{2\pi} \equiv \mathcal{G}_0(\mathbf{r},\mathbf{r}) \simeq \int_0^{1/L_H} \frac{d\mathbf{p}}{(2\pi)^2}\frac{1}{\gamma_+ - \bar{\gamma}_c + L_H^2(\mathbf{p}^2/2 + \alpha g_0)} , \tag{6}$$

where (taking $\omega_D\tau_H \gg 1$) $\bar{\gamma}_c \equiv \ln(\omega_D\tau_H)$, and $\alpha = \kappa/g^2(\Delta_0\tau_H)^2$ characterizes the strength of fluctuations, where κ is some numerical constant. Setting $\alpha = 0$, the self-consistent equation recovers the constant DoS of the ensemble averaged operator. In this approximation, as expected from the AG condition, the transition to the superconducting phase takes place into a state with a spatially homogeneous order parameter when $(\lambda_{\rm BCS}\,\nu)^{-1} - \bar{\gamma}_c \equiv \ln(2/\Delta_0\bar{\tau}_H^c) = 0$. Reinstating mesoscopic fluctuations (through α), one finds the renormalized edge $\gamma_c \simeq \bar{\gamma}_c + \alpha$, valid in the limit $(\gamma_c - \bar{\gamma}_c)/\bar{\gamma}_c \ll 1$, from which one can infer the shift of the critical field,

$$\frac{1}{\tau_H^c} \simeq \frac{1}{\bar{\tau}_H^c}\left(1 + \frac{\kappa}{4g^2}\right) . \qquad (7)$$

An expansion of the DoS in the vicinity of γ_c obtains $\rho(\gamma) = \langle \mathrm{tr}\,\delta(\gamma - \hat{\Pi}/\nu)\rangle_\Pi \simeq (3/\pi^2\alpha)^{1/2}\sqrt{\gamma_c - \gamma}$. Note that inclusion of the $\langle \delta\Pi_0(\mathbf{r},\mathbf{r})\delta\Pi_0(\mathbf{r}',\mathbf{r}')\rangle_V$ correlations leads only to a change in κ.

Therefore, leaving aside the potential for a small exponential band of Lifshitz tail states of $\hat{\Pi}_0$, at the level of the linearised saddle-point approximation, the onset of superconductivity at $T = 0$ takes places at a value of $1/\tau_H^c$ renormalized from that predicted by AG theory. The bulk transition is to a glass-like phase in which the *complex* order parameter inherits the multi-fractal structure implied by the critical theory,

$$\langle |\Delta(\mathbf{r},t)|^2 |\Delta(\mathbf{r}',t')|^2\rangle_{\Pi,\Delta} - \langle |\Delta(\mathbf{r},t)|^2\rangle_{\Pi,\Delta}^2 \sim f(\mathbf{r}-\mathbf{r}') . \qquad (8)$$

The integrity of the saddle-point approximation depends sensitively on the impact of dynamical and spatial fluctuations of the order parameter. Their significance can be inferred from the behaviour of the impurity averaged susceptibility. The gradient expansion, $\langle \Pi_{\omega_m}(\mathbf{r},\mathbf{r}')\rangle_V \simeq \nu\delta(\mathbf{r}-\mathbf{r}')[\gamma_c + L_H^2\partial_{\mathbf{r}'}^2/2 - |\omega_m|\tau_H/4\pi]$, shows the system to be dissipative (both outside and within the gapless ordered phase) with a rate proportional to $1/\tau_H$ implying a dynamical exponent $z = 2$. In the two-dimensional geometry, this places the system at its upper critical dimension [8]. In the presence of mesoscopic fluctuations, the phase space for low-energy fluctuations is only diminished justifying the saddle-point analysis adopted in this work.

At temperatures $T \neq 0$, the thermodynamic properties of the system demand two further considerations: As well as the obvious significance of thermal fluctuations of the order parameter, the mechanisms of quantum interference, on which the long-range correlations of the pairing susceptibility rely, become gradually extinguished. Thermal smearing limits the long-range correlations of $\delta\Pi_0(\mathbf{r},\mathbf{r}')$ to length scales $|\mathbf{r}-\mathbf{r}'|$ smaller than the thermal length $L_T = \sqrt{D/T}$, i.e. $C(\mathbf{r}-\mathbf{r}') \sim (\nu^2/g^2)(\xi_0/|\mathbf{r}-\mathbf{r}'|)^4 e^{-|\mathbf{r}-\mathbf{r}'|/L_T}$. Providing L_T remains greatly in excess of L_H, such dependence motivates a coarse-grained description: Over the range $1/\bar{\tau}_H^c < 1/\tau_H < 1/\tau_H^c$, each domain of size L_T has condensed into a glass-like superconducting droplet, weakly connected to neighboring domains by the residual fluctuations of $\delta\hat{\Pi}_0$. On length scales in excess of L_T, one therefore expects to recover a "gauge glass" picture analogous to that envisaged by Spivak and Zhou [2] with a low-temperature behaviour that can be inferred from the analysis of Galitski and Larkin [3].

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