

Critical velocity of a mobile impurity in one-dimensional quantum liquids

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Impurity motion in one-dimensional quantum fluids has a number of interesting features not present in higher dimensions. Remarkably, the impurity dispersion relation acquires *periodic* dependence on momentum. For a light impurity, inflection points of the latter define a critical velocity below which perfect superfluidity persists at zero temperature. It is shown that for heavier impurities with mass M exceeding a critical mass M_c , the dispersion develops periodic metastable branches, altering the notion of critical velocity. The presence of metastable branches may be detected using an external driving force, under which the impurity climbs metastable branches until it reaches a branch termination point or undergoes a random tunneling event. In its descent, the impurity experiences an abrupt change in velocity and expels an energy loss predicted to lead to a non-analytic dependence of the drift velocity on small forces.

The study of the effect of impurities constitutes a major theme in condensed matter physics. In the solid state impurities are typically regarded as *immobile*, acting as static perturbations. Prime examples include the residual resistance of metals and the Kondo effect. In other (typically liquid) states *mobile* impurities play a vital role, with the influence of ^3He impurities on the transport properties of superfluid ^4He being perhaps the best studied example [1, 2].

The traditional goal has been to understand the effect of foreign particles or defects on an otherwise pure system. However, the converse problem – to determine the effect of the surrounding medium on the motion of impurities – is increasingly relevant to the study of ultracold atomic gases. There are several ways in which mobile impurities appear naturally in these systems: (i) a small fraction of atoms may be transferred to a different hyperfine state, creating a dilute gas of distinguishable impurities of the same mass [3–5]; (ii) mixtures of different atomic species (*e.g.*, ^{87}Rb and ^{41}K in Ref. [6]) with one dilute component give a route to impurities of mass M different from the mass m of the host particles; (iii) *Ions* of Yb^+ , Ba^+ or Rb^+ have been placed in a Bose-Einstein condensate of neutral ^{87}Rb atoms [7, 8]. In each of these situations, impurity dynamics may be investigated with the help of external forces that act selectively on the impurity.

Owing to recent advances in cold atom trapping, systems in reduced dimensionalities are now well within experimental reach [5, 7, 8], bringing to light a wide range of rich and peculiar phenomena. In this Letter we focus on one spatial dimension (1D) where the dispersion $E(P)$ of an impurity moving in a fluid possesses several remarkable features not present in higher dimensions. The most prominent property is its *periodicity*, which can be explained as follows. The motion of N particles on a ring with circumference L results in the *total* momentum being quantized in units of $2\pi\hbar N/L = 2\pi\hbar n$, where n is the one-dimensional density of the fluid. On the

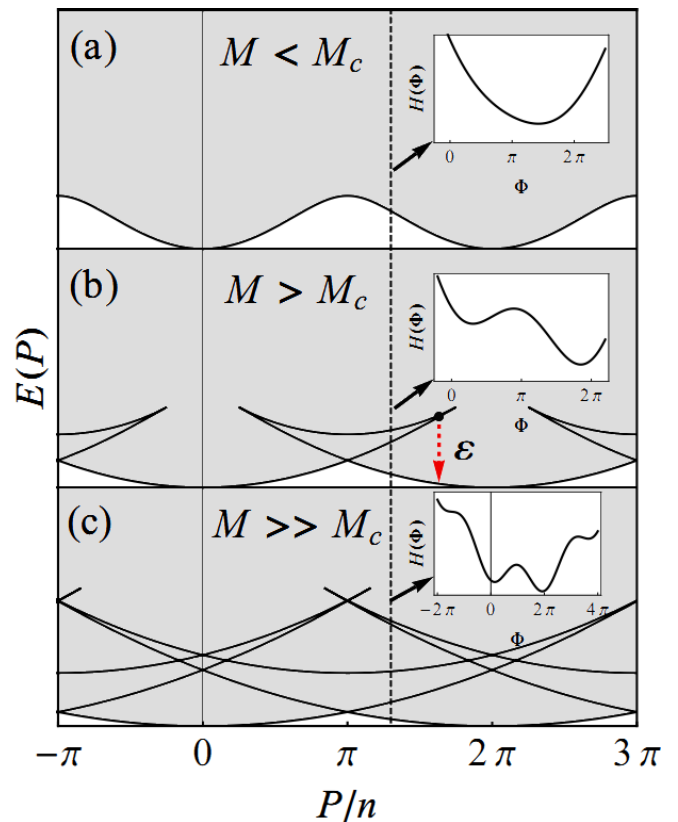


FIG. 1. Dispersion law for an impurity in a 1d quantum fluid for various impurity masses. (a) $M < M_c$, $E(P)$ is a smooth function of P and H (inset, Eq. (2)) has a single minimum. (b) $M \gtrsim M_c$, there exists a metastable minimum. An impurity driven by a force climbs a metastable branch until tunneling or H loses its minimum. Each cycle releases energy ε to the system. (c) $M \gg M_c$, many minima co-exist.

the other hand, in the thermodynamic limit $N, L \rightarrow \infty$, $N/L = n$, this motion costs no kinetic energy due to infinite total mass Nm . This leads to the conclusion that

any multiple of $2\pi\hbar n$ can be added to the momentum of the impurity *at no energy cost* and the dispersion is periodic, $E(P) = E(P + 2\pi\hbar n)$, see Fig. 1. Explicit examples [9, 10] are furnished by models solvable via Bethe Ansatz [11–13]. A dramatic consequence of the dispersion periodicity is the possibility of observing Bloch oscillations of a driven impurity in 1D *in the absence of a lattice* [14, 15].

The same picture of momentum transferred to the fluid at no energy cost leads to the prediction of vanishing Landau critical velocity in 1D [16, 17]. It was shown that an *infinitely massive* object moving with any nonzero velocity with respect to the background nucleates phase slips corresponding to one quantum $2\pi\hbar n$ of total momentum transferred to the fluid. The finite rate of this process can then be related to the energy and momentum dissipation in the liquid, precluding superfluidity. On the other hand, a mobile impurity with *finite mass* M can propagate without dissipation at zero temperature with velocity given by slope of the dispersion $V = \partial E / \partial P$. Hence, the maximum slope of the dispersion defines a non-zero critical velocity for light mobile impurities.

The main result of this work is the prediction that the above regimes of heavy and light impurities are separated by a quantum phase transition taking place at a critical mass M_c , given by Eq. (3). We show that above the critical mass the many-body ground state in a total momentum sector P has cusps at $P = \pm\hbar\pi n, \pm 3\hbar\pi n, \dots$, while the impurity dispersion develops a swallowtail structure with metastable branches shown in Fig. 1. In this case the true *thermodynamic* critical velocity \mathcal{V}_c , given by the maximal slope of the many-body ground state, should be distinguished from the *dynamic* critical velocity V_c obtained from the maximal slope of the metastable branch. It can be shown that for $M \gg M_c$ the thermodynamic critical velocity is inversely proportional to M consistent with the infinite mass limit of Refs. [16, 17].

Metastability and two definitions of critical velocity. For a concrete example of how a metastable impurity dispersion can arise, let us take our fluid to be a weakly interacting Bose gas. At the mean field level, appropriate for weak interactions, the condensate wavefunction develops a phase drop Φ across the impurity located at X . In the limit of strong coupling between gas and impurity, the latter gives rise to the Josephson term

$$H_d(\Phi) = -nV_c \cos \Phi \quad (1)$$

(we have set $\hbar = 1$) in the impurity energy. Here V_c is the dynamical critical velocity which depends on the details of the impurity. The phase drop inevitably creates a background supercurrent contributing a term $n\Phi$ to the *total* momentum P . The total energy of the impurity and the background can then be written as

$$H(P, \Phi) = \frac{1}{2\mathcal{M}}(P - n\Phi)^2 + H_d(\Phi) \quad (2)$$

The first term is the kinetic energy of the impurity moving with velocity $V = (P - n\Phi)/\mathcal{M}$, where \mathcal{M} is the total

mass of the impurity and induced depletion cloud. As we shall see one can assume $\mathcal{M} \simeq M$ for a sufficiently heavy impurity.

The phase drop Φ represents a collective coordinate characterizing the state of the impurity equilibrated with the background [15]. Its value is determined from the requirement of the minimum of the total energy (2) leading to the matching $P - n\Phi = MV_c \sin \Phi$ of the Josephson current $nV_c \sin \Phi$ to the current nV experienced by the impurity in its rest frame. For sufficiently large mass $M > M_c$ this equation has several solutions corresponding to different arrangements of the impurity and the background. The total energy $H(P, \Phi)$ in Eq. (2) can therefore have minima for several values of the phase drop Φ for a given value of the total momentum P . Plotting the corresponding energies results in a typical swallowtail structure for the dispersion, see Fig. 1.

Following the global minimum $E_-(P)$ of the dispersion one sees that it develops cusps at the crossing points $P = \pm\pi n, \pm 3\pi n, \dots$ corresponding to a first order transition as a function of the total momentum P . In addition to the true ground state $E_-(P)$ there is also a metastable branch $E_+(P)$ corresponding to local minimum. The minima corresponding to $E_-(P)$ and $E_+(P)$ are separated by the maximum. At the termination points P_t the maximum merges with the local minimum at $\Phi = \Phi_t$ and the metastable branch ceases to exist. The critical mass M_c for the swallowtail catastrophe can be estimated from the simple model Eq. (2) as

$$M_c = \pi n / V_c = Km(c/V_c), \quad (3)$$

where $K = \pi n / mc$ is the Luttinger parameter of the liquid depending on the speed of sound c . For a strongly repulsive impurity in a weakly interacting background $M_c \gg m$ since $K \gg 1$ and $c \gg V_c$, justifying *a posteriori* the assumption $\mathcal{M} \simeq M$.

For $M \gg M_c$ the termination point corresponds to $\sin \Phi_t = \pm 1$ and the mass-independent velocity $\partial E_+ / \partial P = \pm V_c$. In contrast, the thermodynamic critical velocity, defined with the help of the stable branch $\mathcal{V}_c = \partial E_- / \partial P$, behaves as $1/M$ resulting in zero critical velocity for infinitely heavy impurity [16, 17].

Quantum fluctuations can smear the transition by producing a mechanism for the decay of the metastable branch [10]. The metastable branch $E_+(P)$ is well defined as long as the decay rate $\Gamma(P)$ to the lower branch satisfies $\Gamma(P) \ll E_+(P) - E_-(P) \approx 2V_c|P - \pi n|$. We show below that $\Gamma(P)$ vanishes as a power law near the crossing points, $\Gamma(P) \propto |P - \pi n|^\alpha$ for $P \rightarrow \pi n$ with the exponent α given in Eq. (8). Self-consistency therefore requires $\alpha > 1$, with $\alpha = 1$ being the criterion for the transition between the two classes of dispersion curve [10].

Before proceeding we note that the qualitative features discussed above in no way depend upon the specific model Eq. (1) and occur (for example) in the dispersion relation of a particle moving in a Bose gas described by

the Gross–Pitaevskii equation, valid for arbitrary coupling between the gas and impurity [18].

Depletion model. We now introduce a framework to describe the periodic impurity dispersion, the metastable branch, and its decay in the general case. The moving depleton, *i.e.* impurity dressed by the liquid depletion, is characterized by a number of particles N expelled from its core in addition to the phase drop Φ . The existence of the two collective slow coordinates is associated with the presence of the two conservation laws: number of particles and momentum [15]. The depleton’s dynamics is then specified by the following Hamiltonian

$$H(P, \Phi, N) = \frac{1}{2\mathcal{M}}(P - n\Phi)^2 + \mu N + H_d(\Phi, N). \quad (4)$$

Here H_d is the Φ -periodic energy function of the depletion cloud, μ is the chemical potential of the liquid in the absence of the impurity and $\mathcal{M} = M - mN$. In the limit of strongly repulsive impurity the dynamics of N is frozen and $H_d(\Phi, N)$ reduces to the simple Josephson form Eq. (1). If the mass \mathcal{M} is sufficiently large, the Hamiltonian (4) (being almost a periodic energy function of Φ) possesses many metastable minima in addition to the absolute one.

We wish to determine the tunneling rate from the above mentioned metastable minima to the ground state branch at the same momentum P . Such tunneling is accompanied by a change in the phase drop $\Delta\Phi$ and number of depleted particles ΔN , despite the momentum of both states being equal [19]. Therefore one must know the Lagrangian governing the dynamics of the collective variables Φ and N . As shown in Ref. [15], this requires the introduction of, and coupling to, the phonon subsystem. The latter may be described by small deviations of the density, $\rho(x, t)$, and velocity, $u(x, t)$, fields from their unperturbed values. It is convenient to introduce the superfluid phase $\varphi(x, t)$ and the displacement field $\vartheta(x, t)$ defined by $u = \partial_x \varphi / m$ and $\rho = \partial_x \vartheta / \pi$.

The effective action governing these harmonic degrees of freedom is that of the Luttinger liquid [20, 21]

$$S_{\text{LL}} = \frac{1}{\pi} \int dt dx \left[-\partial_x \vartheta \partial_t \varphi - \frac{c}{2K} (\partial_x \vartheta)^2 - \frac{cK}{2} (\partial_x \varphi)^2 \right], \quad (5)$$

while their coupling to the depleton degrees of freedom takes a universal form,

$$S_{\text{int}} = \int dt \left[-\dot{\Phi} \vartheta(X, t) / \pi - \dot{N} \varphi(X, t) \right]. \quad (6)$$

It is the phonon subsystem which must supply the macroscopic momentum $n\Delta\Phi$ and take away ΔN particles from the depleton. Near the point $P = \pi n$, the phonons have only a small amount of energy ε with which to do this. This is low-probability event, or ‘under-the-barrier’ evolution of the phononic system, which can be described in imaginary time by performing the Wick rotation $t \rightarrow -i\tau$ in Eqs. (5),(6).

We assume the tunneling evolution occurs for an imaginary time $\bar{\tau}$, during which $\dot{\Phi} = (\delta(\tau) - \delta(\tau - \bar{\tau})) \Delta\Phi$ and $\dot{N} = (\delta(\tau) - \delta(\tau - \bar{\tau})) \Delta N$. By performing the standard Gaussian integration over the phonon degrees of freedom, one arrives at the tunneling action $\bar{S}(\bar{\tau}) = (\alpha_+ + \alpha_-) \ln(c\bar{\tau}/l)$, where $l \gtrsim (mc)^{-1}$ is a short distance cut-off beyond which the point-particle description of the depleton breaks down. The coefficients α_{\pm} are given by

$$\alpha_{\pm} = K \left[\frac{\Delta\Phi}{2\pi} \pm \frac{\Delta N}{2K} \right]^2. \quad (7)$$

Equation (7) was obtained previously in Ref. [22] in the context of the ‘photo-solitonic’ effect. Here we show that it is true in the context of a generalized impurity, of which the dark soliton is only one example.

The tunneling rate at energy ε is then calculated with the help of the Fourier transformation

$$\Gamma(\varepsilon) \sim \int d\bar{\tau} e^{-\bar{S}(\bar{\tau}) - i\varepsilon\bar{\tau}} \sim \varepsilon^{\alpha}; \quad \alpha = \alpha_+ + \alpha_- - 1. \quad (8)$$

Writing $\varepsilon = 2\mathcal{V}_c(P - \pi n)$ gives the momentum dependent tunneling rate advertised earlier.

The logarithmic behavior of the tunneling action is valid only on the long time-scale $\bar{\tau} \gtrsim l/c$, implying the validity of Eq. (8) is restricted to the regime of small ε close to $P = \pi n$. In such a case the velocities of the upper and lower branches of the dispersion are essentially opposite and close to $\pm\mathcal{V}_c$, while the change in number of particles ΔN , being an even function of velocity, is negligible. As a result one arrives at the exponent $\alpha = 2K(\Delta\Phi/2\pi)^2 - 1$. This single collective coordinate limit, *i.e.* $\Delta N = 0$, agrees with the previous result of Ref. [23] (cf. Eq. (7.20) therein) based on a phenomenological model of tunneling between two minima in the presence of coupling to an environment.

Higher order processes involve tunneling back and forth between the two minima that become degenerate for $\varepsilon = 0$, leading to an effective spin-boson or anisotropic Kondo description [10]. The inclusion of such processes is only necessary if $\alpha < 1$, meaning that tunneling is a relevant perturbation. This leads to the rounding out of the cusp in the dispersion $E_-(P)$ at $P = \pi n$, and the disappearance of the metastable branch. Thus $\alpha = 1$ is the exact condition determining the critical mass $M = M_c$.

In the infinite mass limit with fixed velocity $V = P/M$ Eq. (4) becomes essentially a washboard potential, Fig. 1c, $H \xrightarrow{M \rightarrow \infty} nV\Phi + H_d(\Phi)$ (change ΔN between adjacent minima goes to zero in this limit). Thus the change $\Delta\Phi$ between the adjacent minima is exactly 2π , while the energy difference is $\varepsilon = 2\pi nV$. As a result, one finds $\alpha = 2K - 1$ and the power law dependence of the tunneling rate $\Gamma \sim V^{2K-1}$ [24] on the velocity in full agreement with the result of Ref. [16, 17].

In the limit of large energy detuning, *i.e.*, away from $P = \pi n$, the tunneling rate does not take the form (8). In this region the barrier is small and the impurity emerges

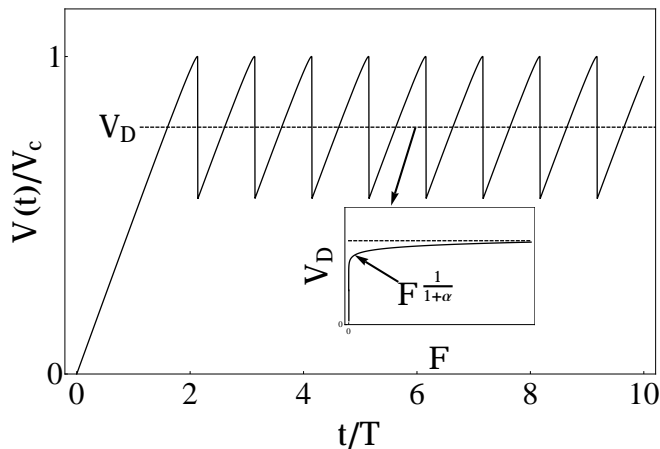


FIG. 2. Impurity velocity (solid) as a function time. The average velocity is denoted by V_D (dashed, inset), while the oscillation amplitude V_B is measured between the maximum and minimum of $V(t)$ (dotted). Inset: V_D as a function of the applied force. Due to quantum tunneling near $P = \pi n$, the drift velocity goes to zero as a power law with exponent $1/(\alpha + 1)$ (solid) given by Eq. (10) for $F < F_c$ and saturates to the semi-classical result for $F > F_c$.

from under it with only slightly less energy, despite the large detuning between the branches (see Fig. 1 insets). Therefore most of the descent from the upper branch corresponds to viscous relaxation in real time and not an under-the-barrier tunneling trajectory. Thus, near the termination momentum $P_t \simeq MV_c$, the potential may be approximated by a cubic polynomial. The dissipative tunneling through such a potential was described in Ref. [25], leading to the escape rate $\Gamma \propto e^{-S}$, where

$$\bar{S} = 2K \frac{P_t - P}{\sqrt{M^2 V_c^2 - n^2}}. \quad (9)$$

We see that for both large and small energy detuning, Eqs. (8), (9) the tunneling action scales with the Luttinger parameter K .

One way to explore the metastable branches of the dispersion is by applying an external force F to the impurity and studying the ensuing dynamics. For sufficiently strong force the driven impurity overshoots the ground state branch and follows the metastable branch until it either tunnels or reaches the termination point P_t (see Fig. 1). The energy dissipated per cycle, ε , must be supplied by external force: $FV_D = \varepsilon/T$ resulting in the drift velocity $V_D = \varepsilon/2\pi n$. Here $T = 2\pi n/F$ is the period of the motion.

Tunneling is negligible when $\Gamma T \ll 1$, which occurs if the force is sufficiently large. In such a case the external force drives the impurity all the way up to the termination point P_t . It then loses the energy minimum and relaxes to the lower branch, emitting phonons. As

a result, the energy dissipated per cycle and thus drift velocity are essentially independent of the applied force. This is in stark contrast to the case of $M < M_c$, where the phonon emission due to dipole radiation gives rise to a linear in F drift velocity and hence a *finite* impurity mobility $\sigma = V_D/F$ [15].

The impurity velocity takes the form of a saw-tooth trajectory, experiencing a slow build-up and a sudden drop when the velocity reaches V_c , see Fig. 2. As a function of the mass, V_D saturates to the critical velocity in the limit $M \rightarrow \infty$. This is most easily seen by noting that in such a limit the energy drop occurs between two adjacent and essentially parallel parabolas: $\varepsilon = 2\pi n V_D = P^2/2M - (P - 2\pi n)^2/2M \rightarrow 2\pi n V_c$. On the same grounds, the amplitude of the saw-tooth oscillations is a decreasing function of M going to zero in the infinite mass limit.

Quantum effects become dominant only if the time spent on a metastable branch is comparable with the inverse tunneling rate: $\Gamma T \gtrsim 1$. Thus, the effect of quantum tunneling plays a role only for very small forces. In this limit, the impurity is likely to tunnel at a momentum immediately above $P = \pi n$, right when it enters the metastable branch. Using the rate Eq. (8) to calculate the tunneling-averaged energy drop $\langle \varepsilon \rangle$, we find for the drift velocity [18]

$$V_D = \frac{c}{K} \left(\frac{F}{F_c} \right)^{\frac{1}{\alpha+1}}. \quad (10)$$

The non-analytic dependence of the drift on small forces is the result of the cusp of the ground state energy at $P = \pi n$ and implies the divergence of the mobility $\sigma = V_D/F$ as $F \rightarrow 0$ across the $M = M_c$ transition (see inset of Fig. 2). We thus find, as expected, that the adiabatic theorem is ultimately upheld: the energy dissipated, and hence V_D , goes to zero with the force.

In conclusion, we have shown that the dispersion relation of a mobile impurity undergoes a sudden change when the mass of the impurity exceeds a certain critical value. The latter depends on the interaction parameter within the host liquid and thus may be varied experimentally. The predicted transition can be probed by applying an external force and monitoring either impurity velocity or excitations of the host liquid. The change of dispersion also affects the oscillation frequency and dissipation of the impurity in the trapping potential. With investigations of impurity motion in 1D atomic gases now underway [5, 7, 8], we hope that these predictions will guide future experiments.

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