PHASE COHERENCE PHENOMENA IN DISORDERED SUPERCONDUCTORS

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1. Introduction

Mechanisms of quantum phase coherence heavily influence spectral and transport properties of weakly disordered normal conductors. Such effects are manifest in weak and strong localization effects, and characteristic fluctuation phenomena. Over the past thirty years, theoretical progress in elucidating the mechanisms of quantum phase coherence in weakly disordered structures has been substantial: By now a consistent theory of weakly interacting disordered structures has been developed (For a review, see e.g., Refs. [1–3]).

At the same time, considerable experimental effort has been directed towards the exploration of the influence of phase coherence effects on the quasi-particle properties of disordered superconductors. Again, attempts to develop a consistent theory have enjoyed great success. By now a reliable theory of the weakly interacting superconducting system has been formulated. Yet, a complete description of the phenomenology of the disordered superconductor in the presence of strong interaction effects has yet to be established. The continuing developments and refinements of experimental techniques continue to present fresh challenges to theoretical investigations.

On this background, the aim of these lecture notes is to selectively review the recent development of a quasi-classical field theoretic framework to describe phase coherence phenomena in disordered superconductors. This approach, which is motivated by the parallel formulation of the theory of the normal disordered system, presents average properties of the superconductor as a quantum field theory with an action of non-linear σ-model type. The limited scope of these lectures does not permit an extensive review the many applications of this technique. Instead, to illustrate the impact of quantum phase coherence phenomena on the quasi-particle properties of the disordered superconducting system, and the practical application of the field theoretic scheme, the final part of these notes will be devoted to a study of the magnetic impurity system.

Before turning to the construction of the field theoretic scheme, we will begin these notes with a qualitative discussion of phase coherence phenomena in the superconducting environment placing emphasis on the importance of fundamental symmetries. To close the introductory section, we will outline the quasi-classical theory which forms the basis of the field theoretic scheme. In section 2 we will develop a quantum field theory of the weakly disordered non-interacting superconducting system (i.e. in the mean-field BCS approximation). To illustrate a simple application of this technique, we will explore the spectral properties of a normal quantum dot contacted to a superconducting terminal. Finally, in section 3, we will present a detailed study of the influence of magnetic impurities in the disordered superconducting system. This single application will emphasize a number of generic features of the phase coherent superconducting system including unusual spectral and localization properties and the importance of effects non-perturbative in the disorder.

To orient our discussion, however, let us first briefly recapitulate the BCS mean-field theory of superconductivity in order to establish some notations and definitions.

1.1. THE BCS THEORY

In the mean-field approximation, the second quantized BCS Hamiltonian of a weakly disordered superconductor is defined by

$$\hat{H}_{\rm bcs} = \int d\mathbf{r} \Big[\sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_\sigma(\mathbf{r}) \left(\frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}/c)^2 + W(\mathbf{r}) - \epsilon_F \right) \psi_\sigma \\ + \Delta(\mathbf{r})\psi^\dagger_\uparrow(\mathbf{r})\psi^\dagger_\downarrow(\mathbf{r}) + \Delta^*(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}) \Big] \tag{1}$$

where $\psi^\dagger_\sigma(\mathbf{r})$ creates an electron of spin σ at position $\mathbf{r}$, ϵ_F denotes the Fermi energy, $\mathbf{A}$ represents the vector potential of an external electromagnetic field, and $W(\mathbf{r})$ an impurity scattering potential. The order parameter is determined self-consistently from the condition $\Delta(\mathbf{r}) = -(\lambda/\nu)\langle\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle$, where λ is the (dimensionless) BCS coupling constant and ν represents the average density of states (DoS) per spin of the normal system.[1] Defining the Bogoliubov transform

$$\psi_\uparrow(\mathbf{r}) = \sum_i \left[\gamma_{i\uparrow} u_i(\mathbf{r}) - \gamma^\dagger_{i\downarrow} v^*_i(\mathbf{r})\right], \qquad \psi_\downarrow(\mathbf{r}) = \sum_i \left[\gamma_{i\downarrow} u_i(\mathbf{r}) + \gamma^\dagger_{i\uparrow} v^*_i(\mathbf{r})\right]$$

the Hamiltonian can be brought to a diagonal form by choosing the spinor elements $u_\alpha(\mathbf{r})$ and $v_\alpha(\mathbf{r})$ to satisfy the coupled Bogoliubov-de Gennes (BdG)

[1] To avoid ambiguity, this is be the density of states per d-dimensional volume, for an effectively d-dimensional system

equations

$$\begin{aligned} \hat{H}u_\alpha(\mathbf{r}) + \Delta(\mathbf{r})v_\alpha(\mathbf{r}) &= E_\alpha u_\alpha(\mathbf{r}) \\ -\hat{H}^* v_\alpha(\mathbf{r}) + \Delta^*(\mathbf{r})u_\alpha(\mathbf{r}) &= E_\alpha v_\alpha(\mathbf{r}), \end{aligned} \tag{2}$$

with eigenvalue E_α. Here $\hat{H} = \hat{H}_0 + W$ represents the particle Hamiltonian of the normal system with $\hat{H}_0 = (\hat{\mathbf{p}} - (e/c)\mathbf{A})^2/2m - \epsilon_F$. Since u_α and v_α are eigenfunctions of a linear operator, the spinor wavefunction $\phi_\alpha^T = (u_\alpha, v_\alpha)$ can be normalized according to $\int d\mathbf{r}\ \phi_\alpha^\dagger(\mathbf{r}) \cdot \phi_\alpha(\mathbf{r}) = 1$. Moreover, the functions u_α and v_α form a complete basis such that $\sum_\alpha \phi_\alpha(\mathbf{r}) \otimes \phi_\alpha^\dagger(\mathbf{r}') = \mathbb{1}^{\rm ph}\delta^d(\mathbf{r} - \mathbf{r}')$. Using this expression, we can define the advanced and retarded Gor'kov Green function as

$$\hat{G}^{\rm r,a}_{\rm Gorkov} = (\epsilon \pm i0 - \hat{H}_{\rm Gorkov})^{-1}$$

where the quasi-particle Gor'kov Hamiltonian takes the form

$$\hat{H}_{\rm Gorkov} = \begin{pmatrix} \hat{H} & \Delta \\ \Delta^* & -\hat{H}^T \end{pmatrix}. \tag{3}$$

Of particular interest later will be the quasi-particle density of states (DoS) per one spin projection obtained from the relation[2]

$$\nu(\epsilon) = \frac{1}{\pi}{\rm tr\ Im}\ \hat{G}^{\rm a}_{\rm Gorkov}(\epsilon) = \sum_i \delta(\epsilon - E_\alpha).$$

In terms of the Gor'kov Green's function the self-consistency equation is

$$\Delta(\mathbf{r}) = -\frac{\lambda}{\nu} T \sum_{\epsilon_n} \left(\hat{G}_{\rm Gorkov}(\epsilon_n)\right)_{12} (\mathbf{r}, \mathbf{r}) \ , \tag{4}$$

where the Matsubara Green function $G_{\rm Gorkov}(\epsilon_n)$ can be found from the analytical property $\hat{G}(\epsilon_n) = \hat{G}^{\rm a}(i\epsilon_n)$ for $\epsilon_n < 0$, and $\epsilon_n = \pi T(2n+1)$ denotes the set of fermionic Matsubara frequencies.

To explore the influence of disorder it is important to understand the fundamental symmetries of the Hamiltonian. Introducing Pauli matrices $\sigma_i^{\rm ph}$ which operate in the matrix or ph-sector of $\hat{H}_{\rm Gorkov}$, the quasi-particle Hamiltonian exhibits the ph-symmetry

$$\hat{H}_{\rm Gorkov} = -\sigma_2^{\rm ph} \hat{H}^T_{\rm Gorkov} \sigma_2^{\rm ph}. \tag{5}$$

[2] This is the true spectral DoS of the Gor'kov Hamiltonian (3), thus with $\Delta = 0$ it is *twice* the normal metal DoS. Of course, the physical DoS of single-particle excitations is not doubled — these are created by the operator $\gamma_\alpha^\dagger$. The relation to even the simplest measurable quantities — such as the tunneling I-V characteristic — requires a discussion of the coherence factors u_α and v_α [4]. The present definition is chosen to emphasize the universality of expressions we will encounter later.

In the absence of an external vector potential $\mathbf{A}$, a gauge can be specified in which the order parameter is real, upon which the *time-reversal symmetry* $\hat{H}^T_{\rm Gorkov} = \hat{H}_{\rm Gorkov}$ is manifest.

1.2. ANDERSON THEOREM AND THE EFFECT OF DISORDER

Anderson [5] explained why the thermodynamic properties of a 'dirty' s-wave superconductor are largely insensitive to the degree of disorder. This can be understood easily within the Gor'kov formalism. Since Anderson's paper, a dirty superconductor has been understood to be a material in which the elastic scattering rate $1/\tau$ is much larger than the superconducting order parameter $|\Delta|$. The strong inequality $1/\tau \gg |\Delta|$ is referred to as the 'dirty limit'. In the dirty limit impurity scattering washes out any gap anisotropy and one can apply the simple BCS model of the previous section with even greater confidence than in the clean case.[3] Then it is clear from (3) that with $\mathbf{A} = 0$ and constant order parameter, the BdG equations can be solved simply in terms of the eigenvalues ϵ_α and eigenstates of the single-particle Hamiltonian $\hat{H}$,

$$E^{\pm}_\alpha = \pm\sqrt{\epsilon^2_\alpha + |\Delta|^2} \,. \tag{6}$$

Thus the DoS of the superconductor is

$$\nu(\epsilon) = \begin{cases} 0 & \epsilon < |\Delta|, \\ 2\nu_{\rm n}\dfrac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} & \epsilon > |\Delta|, \end{cases} ,$$

independently of the amount of disorder (see Fig. 1). Here we use the fact that the normal metallic DoS $\nu_{\rm n}$ is independent of disorder. More generally the average Gor'kov Green's function at coinciding points appearing in Eq. 4 is unchanged, so the transition temperature $T_{\rm c}$ is unaltered, and so on.

The Anderson theorem is a robust explanation of a striking experimental fact. The conclusion is however suspect from a modern perspective — in the limit of very strong disorder one would expect localization of the single-particle eigenstates to affect superconductivity. The key assumption in the above is that the order parameter is independent of position. This leads to the self-consistency equation (at $T = 0$)

$$1 = -\frac{\lambda}{\nu}\int d\epsilon \frac{1}{\sqrt{\epsilon^2 + |\Delta|^2}}\nu(\epsilon, \mathbf{r}) \,,$$

where $\nu(\epsilon, \mathbf{r})$ is the local DoS of the normal system. Anderson's theorem thus requires the replacement $\nu(\epsilon, \mathbf{r}) \to \nu_{\rm n}$. This is a valid approximation *even in the*

[3] Of course, there are high-energy phenomena $\gtrsim |\Delta|$ where specific details of the interaction (phonon spectrum, etc.) are important, but we will not be considering them.

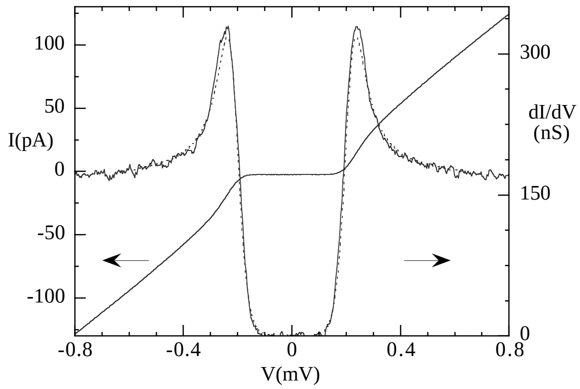

Figure 1. I-V characteristic and differential conductance measured by scanning tunneling microscopy on a superconducting layer of Al at 60mK. The dashed line is a fit using a BCS density of states ($\Delta_{\rm Al} = 210\mu$eV) convoluted with a thermal Fermi distribution (at $T = 210$mK). Taken from Ref. [6].

presence of localization provided that $|\Delta|\nu L_{\rm loc}^d \gg 1$, where $L_{\rm loc}$ is the localization length and d the dimensionality [7]. In fact, the destruction of superconductivity can occur in far more metallic samples due to the dramatic effects of disorder combined with the residual Coulomb interaction. The mean-field treatment of this physics is due to Finkelstein (see e.g. [8]) — but the effects of the Coulomb interaction in dirty superconductors are only well understood in certain limits and not at all generally. Even more surprising is that the BCS model in section 1.1 is compatible with a huge variety of unusual spectral and transport behaviour enabled by novel mesoscopic phase coherence mechanisms.

1.2.1. *Evading the Anderson Theorem*

Thermodynamic properties have not historically been the best place to start looking for mesoscopic effects (it was, for example, a long time before attention was focussed on the persistent currents in normal metals). Spectral properties are the domain of mesoscopics, but the conclusion drawn from Anderson's theorem about the quasi-particle spectrum may appear to preclude any new effects particular to superconducting systems.

In fact the assumptions of Anderson's theorem seem more restrictive today than at the time. The investigation of hybrid electronic devices containing both superconducting (S) and normal (N) metallic elements is an extremely active field of research. Here the order parameter is not constant throughout the system and Anderson's theorem does not apply. At the very least one needs a formulation of the Gor'kov theory capable of handling this spatial inhomogeneity. We will come to this *quasi-classical* description presently. Beyond this description — which dates back to the late 60s — SN systems do in fact exhibit a wide range of novel

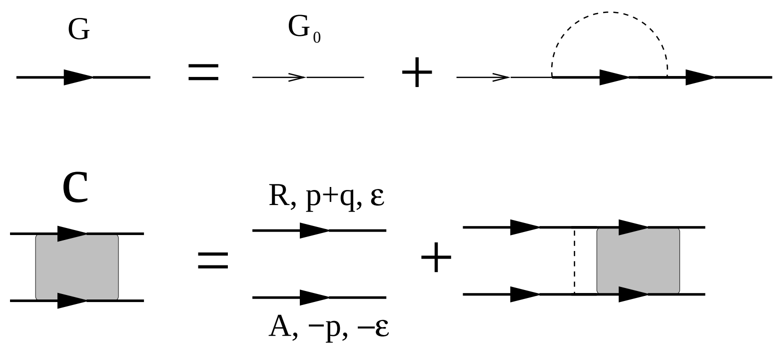

Figure 2. Diagrams for the evaluation of the Cooperon.

mesoscopic phenomena. These are mediated by Andreev [9] reflection — the phase coherent inter-conversion of electrons and holes at the SN interface due to the spectral gap of the bulk superconductor.

We will be concerned only tangentially with hybrid structures in later chapters, so a qualitative description of these effects here is not appropriate (for a discussion, see [10]). There are many other ways, however, to avoid Anderson's conclusion even in a 'bulk' superconductor (including thin films and wires). An important second strand of experimental evidence discussed in Anderson's paper relates to the deleterious effect of *magnetic* impurities on superconductivity. Unconventional superconductors with non s-wave pairing (the high-T_c materials being the most prominent examples) are likewise affected by normal disorder. All these counter-examples have very recently been shown to display dramatic mesoscopic behaviour. We will come to this through a fuller explanation of the robustness to disorder in the conventional s-wave case.

1.3. PAIR PROPAGATION AND THE COOPERON

Within the Gor'kov formalism outlined in section 1.1, an estimate for T_c can be determined by linearizing the self-consistent equation (4) in Δ

$$\begin{aligned}\Delta(\mathbf{r}) &= -\frac{\lambda}{\nu}T\sum_{\epsilon_n}\int d\mathbf{r}'\Delta(\mathbf{r}')\hat{G}_{i\epsilon_n}(\mathbf{r},\mathbf{r}')\hat{G}_{-i\epsilon_n}(\mathbf{r},\mathbf{r}') \qquad (7)\\ &= -\frac{\lambda}{\nu}\int d\mathbf{r}'\int\frac{d\epsilon}{2\pi}\tanh\left(\frac{\epsilon}{2T}\right)\mathrm{Im}\,\hat{G}^{\mathrm{r}}_{\epsilon}(\mathbf{r},\mathbf{r}')\hat{G}^{\mathrm{a}}_{-\epsilon}(\mathbf{r},\mathbf{r}')\ ,\end{aligned}$$

where $\hat{G}_{\epsilon_n}$ is the Green's function corresponding to the single-particle Hamiltonian $\hat{H}$ at imaginary frequency and $\hat{G}^{\mathrm{r,a}}_{\epsilon}$ the real frequency advanced and retarded counterparts. Taking Δ to be constant as before we average over disorder configurations to find $\langle\hat{G}^{\mathrm{r}}_{\epsilon}(\mathbf{r},\mathbf{r}')\hat{G}^{\mathrm{a}}_{-\epsilon}(\mathbf{r},\mathbf{r}')\rangle$. The evaluation may be performed using

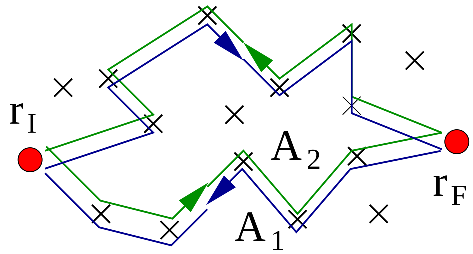

Figure 3. Dominant contributions to time-reversed pair propagation in the Feynman picture. The phase of the amplitude A_1 is the opposite of A_2 if time-reversal symmetry is preserved.

the standard 'cross' technique [11] based on a Gaussian δ-correlated impurity distribution,

$$\langle W(\mathbf{r})\rangle = 0, \qquad \langle W(\mathbf{r})W(\mathbf{r}')\rangle = \frac{1}{2\pi\nu\tau}\delta^d(\mathbf{r}-\mathbf{r}')\ , \tag{8}$$

and is illustrated in Fig. 2. The result is [12]

$$\langle \hat{G}^{\mathrm{r}}_{\epsilon}(\mathbf{r},\mathbf{r}')\hat{G}^{\mathrm{a}}_{-\epsilon}(\mathbf{r},\mathbf{r}')\rangle = \left(\frac{2\pi\nu}{Dq^2-2i\epsilon}\right)_{\mathbf{rr}'}\ . \tag{9}$$

Here $D = v_F^2\tau/d$ is the diffusion constant, where $v_F = p_F/m$ denotes the Fermi velocity. The two-particle quantity under consideration evidently relates to the propagation of a pair of electrons between two points in opposite directions. The diffusion pole structure of the average signals the presence of a hydrodynamic mode of pair propagation known as the *Cooperon*. In the language of the Feynman path integral, this is because the dominant trajectories for the propagation of the pair through a given disorder realization come from the the electrons tracing out precisely time-reversed paths, so that the phase accumulated in the overall amplitude in propagation is completely canceled (see Fig. 3). The phase of of a single propagating electron is scrambled after a time $\sim\tau$, but two particle averages like the above depend on the 'bulk' property D. Their inclusion in diagrammatic calculations typically leads to anomalously large contributions from long wavelengths due to their diffusive structure.

Returning to the matter of determining T_{c}, from the result above, the self-consistency condition (7) takes the form

$$1 = -\lambda \int d\epsilon\,\tanh\left(\frac{\epsilon}{2T}\right)\frac{1}{2\epsilon}\ , \tag{10}$$

independent of disorder, yielding $T_{\mathrm{c}} \sim \omega_{\mathrm{D}}\exp(1/\lambda)$, with ω_{D} the Debye frequency at which the interaction is cut off. The multiple scattering between time-reversed electrons summarized by (7) is absolutely indifferent to the disorder potential through which they propagate. Thus we see the intimate connection between time-reversal invariance in the original single-particle Hamiltonian and Anderson's theorem.

What happens if time-reversal symmetry is broken (by the application of a magnetic field, for example)? Then the propagating pair progressively loses relative phase coherence as time passes. The Cooperon ceases to be a hydrodynamic mode

$$\langle G_{i\epsilon_n}(\mathbf{r},\mathbf{r}')G_{-i\epsilon_n}(\mathbf{r},\mathbf{r}')\rangle = \left(\frac{2\pi\nu}{Dq^2+2|\epsilon_n|+1/\tau_\varphi}\right)_{\mathbf{rr}'} .$$

Here $1/\tau_\varphi$ represents some rate characteristic of the symmetry-breaking perturbation. Substituting this into (7) one obtains the celebrated result obtained by Abrikosov and Gor'kov [13],

$$\ln\left(\frac{T_{\mathrm{c}\,0}}{T_{\mathrm{c}}}\right) = \psi\left(\frac{1}{4\pi\tau_\varphi T_c}+\frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) , \tag{11}$$

where $T_{\mathrm{c}\,0}$ is the critical temperature at $1/\tau_\varphi = 0$. The complete destruction of T_{c} is predicted at $1/\tau_\varphi = 1.76T_{\mathrm{c}\,0}$ (see Fig. 4).

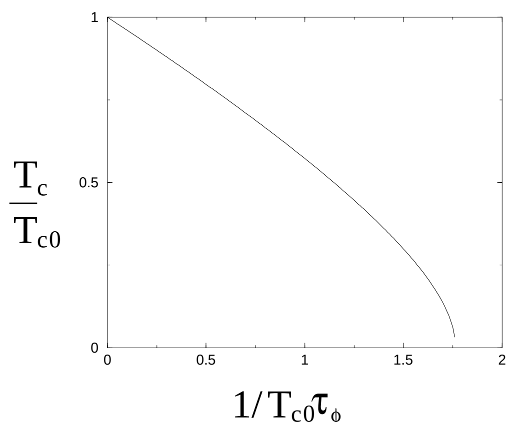

Figure 4. Suppression of T_{c} predicted by the Abrikosov-Gor'kov theory

One of the main themes in the following chapters will be the mesoscopic nature of various processes that impinge on the coherent pair propagation responsible for superconductivity. In this context, we should note that, in addition to the time-reversal symmetry breaking perturbations discussed here, these include both the static and dynamic parts of the Coulomb interaction. While the static part acts like the BCS interaction, the dynamic part like a pair-breaking perturbation. Before we can begin, there is one more subject to introduce.

1.4. SYMMETRIES OF THE HAMILTONIAN AND RANDOM MATRIX THEORY

In the previous section we encountered an important theme in mesoscopics; the central role played by the basic symmetries of the Hamiltonian. In fact there is a limiting sense in which a mesoscopic system is entirely characterized by its symmetries.[4] Let us first focus on the normal system. From the conductivity σ, we can define the conductance $G = \sigma L^{d-2}$ which, making use of the Einstein relation $\sigma = e^2 \nu D$ can be expressed as

$$G = \frac{e^2}{\hbar} \nu L^d \frac{\hbar D}{L^2} = \frac{e^2}{\hbar} g, \qquad g \equiv \frac{E_{\mathrm{T}}}{\delta} \tag{12}$$

where $\delta = 1/\nu L^d$ denotes the average energy level spacing of the normal system, and $E_{\mathrm{T}} = \hbar D/L^2$ represents the typical inverse diffusion time for an electron to cross a sample of dimension L^d — the 'Thouless energy'. This result shows that the conductance of a metallic sample can be expressed as the product of the quantum unit of conductance $e^2/\hbar = (4.1k\Omega)^{-1}$, and a *dimensionless conductance* g equal to the number of levels inside an energy interval E_c. In a good metallic sample, the dimensionless conductance is large, $g \gg 1$.

One of the central tenets of mesoscopic physics is that the spectral properties of Hamiltonian of a disordered electronic system can be modeled as a random matrix of the appropriate symmetry. This remarkable correspondence holds if we are concerned only with energies within E_{T} of the Fermi surface, or equivalently, with times longer than the transport time $t_{\mathrm{D}} = L^2/D$ across the system. Crudely speaking, this is due to the existence of an ergodic regime at these scales when the entire phase space has been explored. If we are only concerned with this regime it is appropriate to take the 'universal' $g \to \infty$ limit. Within the σ-model formalism that will be developed later, the emergence of the random matrix description is very natural.

The random matrix description is formalized by defining a statistical ensemble $P(H)\, dH$ from which the Hamiltonian which models our system will be drawn. The choice encountered most frequently in the literature is the Gaussian ensemble

$$P(H)\, dH = \exp\left[-\frac{1}{v^2}\,\mathrm{tr}\, H^2\right]\, dH\ . \tag{13}$$

Restricting the discussion to ordinary normal metals, three principal universality classes of the Random Matrix Theory (RMT) description can be identified [15] according to whether the matrix H is constrained to be real symmetric ($\beta = 1$, *Orthogonal*), complex Hermitian ($\beta = 2$, *Unitary*), or real quaternion ($\beta = 4$,

[4] In this section we discuss only non-interacting systems (including the mean-field treatment of interactions represented by the Gor'kov Hamiltonian (3)). Recently this has been extended to the interacting case [14]

Symplectic). Hamiltonians invariant under time-reversal belong to the orthogonal ensemble, while those which are not belong to the unitary ensemble. Time-reversal invariant systems with half-integer spin and broken rotational symmetry belong to the third symplectic ensemble.

Expressed in the basis of eigenstates $H = U^\dagger \Lambda U$, where Λ denotes the matrix of eigenvalues, the probability distribution (13) can be recast in the form

$$P(\{\epsilon\})\ d[\{\epsilon\}] = \prod_{i<j} |\epsilon_i - \epsilon_j|^\beta \prod_k e^{-\epsilon_k^2/v^2}\ d\epsilon_k$$

where the invariant measure reveals the characteristic repulsion of the energy levels.

The Dyson classification is made on the basis of the symmetries of time reversal $\mathcal{T}$ and spin rotation $\mathcal{S}$:

$$\mathcal{T} : H = \sigma_2^{\rm sp} H^T \sigma_2^{\rm sp}, \qquad \mathcal{S} : [H, \sigma^{\rm sp}] = 0,$$

where $\sigma_i^{\rm sp}$ are Pauli matrices acting on spin.

In the present context it is natural to ask what happens when we extend the discussion to superconducting systems described by the Gor'kov Hamiltonian. Altland and Zirnbauer [16] have provided the answer, introducing a further *seven* symmetry classes, exhausting the Cartan classification of symmetric spaces upon which they turn out to be based. Their analysis was technical, but we can see the idea through a simple example. As a prototype of the superconducting system let us consider the example of a $2N \times 2N$ matrix Hamiltonian with a particle/hole structure. The simplest case corresponds to $\mathcal{S}$ preserved and $\mathcal{T}$ broken. The Hamiltonian

$$H = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}, \tag{14}$$

where the block diagonal elements are complex Hermitian, $h^\dagger = h$, and the off-diagonal blocks are symmetric, $\Delta^T = \Delta$, exhibits the ph-symmetry

$$H = -\sigma_2^{\rm ph} H^T \sigma_2^{\rm ph}. \tag{15}$$

In this case, according to the Cartan classification scheme, the Hamiltonian (14) belongs to the symmetry class C. Taking the elements to be drawn from a Gaussian ensemble $P(H)\ dH = \exp[-{\rm tr}\ H^2/2v^2]\ dH$, the distribution function takes the general form

$$P(\{\epsilon\}) d[\{\epsilon\}] = \prod_{i<j} |\epsilon_i^2 - \epsilon_j^2|^\beta \prod_k |\epsilon_k|^\alpha e^{-\epsilon_k^2/v^2} d\epsilon_k$$

where $\beta = 2$ and $\alpha = 2$ [16]. The repulsion that the levels feel from $\epsilon = 0$ follows from the privileged place that energy possesses in the Gor'kov Hamiltonian. By imposing the further symmetry of time-reversal (i.e. $h^* = h$ and $\Delta^* = \Delta$), the symmetry is raised to class CI with $\beta = 1$ and $\alpha = 1$. Once again, an extension to a spinful structure identifies two more symmetry classes [17].

Why is the classification scheme useful? In fact, the low-energy, long-ranged properties of the disordered superconducting system are heavily constrained by the fundamental symmetries of the Hamiltonian. We will see that the localization properties of the low-energy quasi-particle states can typically be immediately inferred from the symmetry classification alone.[5]

We saw that the existence of a hydrodynamic Cooperon mode was a fundamental consequence of time-reversal symmetry in a ordinary (non-Gor'kov) Hamiltonian. Therefore the Cooperon should be viewed as a perturbative, finite g, counterpart of the universal RMT description of the orthogonal class. In the same way we can expect that new soft modes will appear as signatures of the new symmetry classes. As their very existence depends on the Gor'kov structure of the Hamiltonian, it is not surprising that the effects of these new modes are singular at low energies. Crudely speaking, the order parameter can be viewed as a potential scattering particle excitations of energy ϵ to hole excitations of energy $-\epsilon$. It is evident that these processes, like the Cooperon, are coherent as $\epsilon \to 0$. Hence the existence of *low energy quasi-particle states* is absolutely necessary for the new channels of interference to be effective. All the aforementioned examples of superconducting systems that evade Anderson's theorem have this property for some parameter ranges and, as such, are candidates for the observation of new mesoscopic effects. For instance, systems of class C symmetry will presumably display some precursor of the level repulsion from $\epsilon = 0$ in the averaged density of states before the universal limit is reached. The possibility of observing dramatic behaviour in *single* quasi-particle properties instead of two-particle properties is an exciting prospect.

This completes our discussion of the phenomenology of the weakly disordered superconducting system. In the following we will develop and apply a field theoretic framework which captures both the perturbative and non-perturbative effects of quantum interference on the quasi-particle properties of the system. However, to prepare our discussion of the field theoretic scheme we begin with a brief review of the quasi-classical theory of superconductivity which forms the basis of this approach.

1.5. THE QUASI-CLASSICAL THEORY

Typically, it is found experimentally that the Fermi energy ϵ_F of a superconductor is always greatly in excess of the order parameter, Δ. In conventional 'low-temperature' superconductors, the ratio ϵ_F/Δ is often as much as 10^3. From

[5] There are rare cases — such as the disordered d-wave superconductor [18, 19] — where the particular nature of the disorder is important.

this fact we can infer that the description of the superconductor in terms of the exact Green function carries with it a certain amount of redundant information. The quasi-classical method exploits this redundancy to develop a simplified theory describing the variation of the Green function on length scales comparable with the coherence length (which, in the clean system, is given by $\xi = v_F/\Delta \gg \lambda_F$). This makes the quasi-classical method ideal for the description of inhomogeneous situations (like the hybrid devices mentioned before).

In the BCS mean-field approximation, the single quasi-particle properties of the superconductor are contained within the equation (of motion) for the advanced Gor'kov Green function (3)

$$\left[\epsilon_- - \hat{\zeta}\sigma_3^{\rm ph} - \hat{\Delta}\right]\hat{G}^{\rm a}_{\rm Gorkov}(\mathbf{r}_1 - \mathbf{r}_2) = \delta^d(\mathbf{r}_1 - \mathbf{r}_2)$$

where $\epsilon_- = \epsilon - i0$, $\hat{\zeta} = \hat{\mathbf{p}}^2/2m - \epsilon_F$, and $\hat{\Delta} = |\Delta|\sigma_1^{\rm ph} e^{-i\varphi\sigma_3^{\rm ph}}$.

In the quasi-classical limit, $\epsilon_F \gg |\Delta|$, fast fluctuations of the Gor'kov Green function (i.e. those at the Fermi wavelength $\lambda_F = 1/p_F$) are modulated by slow variations at the scale of the coherence length $\xi = v_F/\Delta$ of the clean system. In this limit, the important long-ranged information contained within the slow variations of the Gor'kov Green function can be exposed by averaging over the fast fluctuations. Following the procedure outlined in the seminal work of Eilenberger [21], and later by Larkin and Ovchinnikov [22, 23], the resulting equation of motion for the average Green function assumes the form of a kinetic equation

$$v_F \mathbf{n}\cdot\nabla\hat{g}(\mathbf{r},\mathbf{n}) - i\left[\hat{g}(\mathbf{r},\mathbf{n}), (\epsilon_- + \hat{\Delta})\sigma_3^{\rm ph}\right] = 0$$

where, defining $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\zeta = v_F(p - p_F)$, and $\mathbf{n} = \mathbf{p}/p_F$,

$$\hat{g}(\mathbf{r},\mathbf{n}) = \frac{i}{\pi}\sigma_3^{\rm ph}\int d\zeta \overbrace{\int d(\mathbf{r}_1 - \mathbf{r}_2)\hat{G}^-_{\rm Gorkov}(\mathbf{r}_1,\mathbf{r}_2)e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)}}^{\hat{G}^-_{\rm Gorkov}(\mathbf{r},\mathbf{p})}.$$

This Boltzmann-like equation of motion, known as the Eilenberger equation, represents an expansion to leading order in the ratio of λ_F to the scale of spatial variation of the slow modes of the Gor'kov Green function. The Eilenberger Green's function satisfies the non-linear constraint: $\hat{g}(\mathbf{r},\mathbf{n})^2 = \mathbb{1}$, fixed in the usual formulation by the homogeneous BCS solution discussed below [24] (for reasons which will become clear later, we will not dwell here upon the origin of this condition).

In the presence of weak impurity scattering (i.e. $\ell \equiv v_F\tau \gg \lambda_F$), the Eilenberger equation must be supplemented by an additional term which, in the language of the kinetic theory, takes the form of a collision integral. In the Born scattering approximation, the corresponding equation of motion for the average

Green function assumes the form

$$v_F \mathbf{n} \cdot \nabla \hat{g}(\mathbf{r}, \mathbf{n}) - i \left[\hat{g}(\mathbf{r}, \mathbf{n}), (\epsilon_- + \hat{\Delta})\sigma_3^{\mathrm{ph}} \right] = -\frac{1}{2\tau} \left[\hat{g}(\mathbf{r}, \mathbf{n}), \int d\mathbf{n}'\, \hat{g}(\mathbf{r}', \mathbf{n}') \right] .$$

Now, in the dirty limit $\ell \ll \xi$, where $\xi = (D/\Delta)^{1/2}$ represents the superconducting coherence length in the dirty limit, the Eilenberger equation can be simplified further. In this regime the dominant transport mechanism is diffusion. Under these conditions, the dependence of the Green function on the momentum direction ($\mathbf{n} = \mathbf{p}/p_F$) is weak, justifying a moment expansion: $\hat{g}(\mathbf{r}, \mathbf{n}) = \hat{g}_0(\mathbf{r}) + \mathbf{n} \cdot \hat{\mathbf{g}}_1(\mathbf{r}) + \ldots$, where $\hat{g}_0(\mathbf{r}) \gg \mathbf{n} \cdot \hat{\mathbf{g}}_1(\mathbf{r})$. A systematic expansion of the Eilenberger equation in terms of $\hat{\mathbf{g}}_1$ then leads to a nonlinear second-order differential equation — the Usadel equation — for the isotropic component [25],

$$D\nabla \left(g_0(\mathbf{r}) \nabla \hat{g}_0(\mathbf{r}) \right) + i \left[\hat{g}_0(\mathbf{r}), (\epsilon_- + \hat{\Delta})\sigma_3^{\mathrm{ph}} \right] = 0 \,. \tag{16}$$

As in the parent Eilenberger case, the matrix field obeys the non-linear constraint $\hat{g}_0(\mathbf{r})^2 = \mathbb{1}$. Finally, when supplemented by the self-consistent equation for the order parameter,

$$|\Delta(\mathbf{r})| = -\frac{\lambda\pi}{2} T \sum_{\epsilon_n} \mathrm{tr} \left[\sigma_2^{\mathrm{ph}} e^{-i\varphi\sigma_3^{\mathrm{ph}}}\ \hat{g}_0(\mathbf{r}) \right]_{\epsilon = i\epsilon_n} , \tag{17}$$

where the trace runs over the particle/hole degrees of freedom, this equation describes at the mean-field level the quasi-classical properties of the disordered superconducting system. By averaging over the fast fluctuations at the scale of the Fermi wavelength, the long-range properties of the average quasi-classical Green function are expressed as the solution to a non-linear equation of motion.

Let us illustrate the quasi-classical Usadel theory for a weakly disordered bulk singlet superconducting system. In this case, the solution of the mean-field equation can be obtained by adopting the homogeneous parameterization

$$\hat{g}_{\mathrm{bcs}} = \cosh\theta\ \sigma_3^{\mathrm{ph}} - i \sinh\theta\ \sigma_2^{\mathrm{ph}} e^{-i\varphi\sigma_3^{\mathrm{ph}}} . \tag{18}$$

When substituted into Eq. (16), one obtains the homogeneous solution

$$\cosh\theta_{\mathrm{s}} = \frac{\epsilon_-}{E}, \qquad \sinh\theta_{\mathrm{s}} = \frac{|\Delta|}{E} \tag{19}$$

where $E = (\epsilon_-^2 - |\Delta|^2)^{1/2}$. Here the root is taken in such a way that $\lim_{\epsilon\to\infty} E \to \epsilon_-$, i.e. $\theta = 0$. Finally, when the solution (19) is substituted back into the self-consistent equation (17), one obtains the BCS equation for the order parameter,

$$|\Delta| = -\lambda\pi T \sum_{\epsilon_n} \frac{|\Delta|}{(\epsilon_n^2 + |\Delta|^2)^{1/2}} .$$

i.e. at the level of mean-field, the average quasi-classical Green function is insensitive to the random impurity potential — a result compatible with the Anderson theorem.

This concludes our introductory discussion of the disordered superconducting system. The quasi-classical theory (and it's extension to the non-equilibrium systems) has proved to be remarkably successful in explaining mechanisms of phase coherent transport observed in hybrid superconducting/normal compounds. However, as a comprehensive theory, the quasi-classical scheme alone is incomplete: In such environments, low-energy quasi-particle properties become heavily influenced by quantum phase coherence effects not accommodated by the present theory. In the following section, we will develop a description of the superconducting system within the framework of a quantum field theory. Here we will find that the quasi-classical theory above represents the saddle-point of an effective action whose fluctuations encode the missing mechanisms of quantum phase coherence.

2. Field theory of the disordered superconductor

The development of a statistical field theory of the weakly disordered superconductor closely mirrors the formulation of the quasi-classical theory outlined in section 1. However, the benefits of the field theoretic scheme are considerable:

1. Firstly, the field theoretical approach provides a consistent method to explore the influence of mesoscopic fluctuation phenomena both in the "particle/hole" and "advanced/retarded" channels. As discussed above, such effects become pronounced when low-energy quasi-particle states persist. Indeed, such quantum interference effects can be explored even in situations where the mean-field structure is spatially non-trivial such as that encountered with hybrid superconducting/normal structures.
2. Secondly, and more importantly, it provides a secure platform for the further development and analysis of Coulomb interaction effects and non-equilibrium phenomena through straightforward refinements of the field theoretic scheme.
3. Finally, the field theoretic approach has great aesthetic appeal: it's content is largely constrained by the fundamental symmetries of the disordered superconducting system. Within this formulation, the soft low-energy modes responsible for the long-ranged phase coherence properties described in the previous section are exposed.

For these reasons, we will provide a detailed exposition of the field theoretic method from formulation to application. The starting point will be an exact functional integral representation of the generating function of the electron Green function. The latter must be normalized *independently of the disorder*. This can be achieved via the supersymmetry, replica, or Keldysh methods. Since we will restrict attention to the non-interacting system, we will focus on the supersymmetry technique (which extends to the mean-field treatment of superconductivity). In the semi-classical approximation, we will use the intuition afforded by the quasi-classical scheme to identify the low-energy content of the theory of the ensemble averaged system. As a result, we will show that the low-energy, long-ranged properties of the disordered superconductor can be presented as a supersymmetric non-linear σ model.

In the remainder of the chapter we will apply the supersymmetric scheme to analyze the spectral properties of a hybrid superconductor/normal quantum dot device. Later, in the subsequent chapter we will see how this scheme presents a method to explore non-perturbative effects in the magnetic impurity system.

2.1. FUNCTIONAL METHOD

2.1.1. *Generating functional*

To compute the disorder averaged Green function, we will use Efetov's supersymmetry method [26, 27] tailored to the description of the superconducting system [28, 29, 10]. The analysis (and notation) adopted here is based on a pedagogical exposition of the method by Bundschuh, Cassanello, Serban and Zirnbauer [30]. Within the supersymmetric approach, the Gor'kov Green function is obtained from the generating functional[6]

$$\mathcal{Z}[j] = \int D[\bar\psi,\psi]\ \exp\left[\int d\mathbf{r}\ \left(i\bar\psi(\hat H_{\rm Gorkov} - \epsilon_-)\psi + \bar\psi j + \bar j\psi\right)\right],$$

where, as usual, $\epsilon_- \equiv \epsilon - i0$ and, in the mean-field approximation, $\hat H_{\rm Gorkov}$ denotes the Gor'kov Hamiltonian (3). For the moment we ignore the spin structure and retain only the Nambu space. Formally, the infinitesimal, which provides convergence of the field integral, imposes the analytical structure of the Green function. The functional integral is over supervector fields $\psi(\mathbf{r})$ and $\bar\psi(\mathbf{r})$, whose components are commuting and anticommuting (i.e. Grassmann) fields [26]. Introducing both commuting and anticommuting elements ensures the normalization of the field integral, $\mathcal{Z}[0] = 1$ — a trick clearly limited to the mean-field (single quasi-particle) approximation. Thus, in addition to the (physical) particle-hole ($\rm ph$) or Nambu structure, the fields are endowed with an auxiliary "boson-fermion" ($\rm bf$) structure. A generalization to averages over products of Green functions follows straightforwardly by introducing further copies of the field space.

To capture all possible channels of quantum interference in the effective theory is is necessary to further double the field space [27]. This "charge conjugation" (or $\rm cc$) space, is introduced by rearranging the quadratic form of the generating functional as follows:

[6] Historically the field-theoretic approach to disordered electron problems is due to Wegner [31] who used the replica formalism for the derivation of the nonlinear sigma model.

$$\begin{aligned}
&2\bar{\psi}(\hat{H}_{\rm Gorkov} - \epsilon_-)\psi \\
&\quad = \bar{\psi}(\hat{H}_{\rm Gorkov} - \epsilon_-)\psi + \psi^T(\hat{H}^T_{\rm Gorkov} - \epsilon_-)\bar{\psi}^T \\
&\quad = \bar{\psi}(\hat{H}_{\rm Gorkov} - \epsilon_-)\psi + \psi^T(-\sigma_2^{\rm ph}\hat{H}_{\rm Gorkov}\sigma_2^{\rm ph} - \epsilon_-)\bar{\psi}^T \\
&\quad = \bar{\Psi}(\hat{H}_{\rm Gorkov} - \epsilon_-\sigma_3^{\rm cc})\Psi
\end{aligned}$$

where

$$\bar{\Psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\psi} & -\psi^T\sigma_2^{\rm ph}\end{pmatrix}, \qquad \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi \\ \sigma_2^{\rm ph}\bar{\psi}^T\end{pmatrix}.$$

Here the superscript T denotes the supertransposition operation,[7] and $\sigma_i^{\rm cc}$ represent Pauli matrices acting in the charge conjugation space. As a consequence, the two supervector fields $\bar{\Psi}$, and Ψ are not independent but obey the symmetry relations

$$\Psi = \sigma_2^{\rm ph}\gamma\,\bar{\Psi}^T, \qquad \bar{\Psi} = -\Psi^T\,\sigma_2^{\rm ph}\gamma^{-1}, \tag{20}$$

where

$$\gamma = \mathbb{1}^{\rm ph} \otimes \begin{pmatrix}\sigma_1^{\rm cc} & \\ & -i\sigma_2^{\rm cc}\end{pmatrix}_{\rm bf} \tag{21}$$

To summarize, the generating functional for averages of products of Green functions can be written as

$$\mathcal{Z}[0] = \int D[\bar{\Psi},\Psi]\ \exp\left[i\int d\mathbf{r}\ \bar{\Psi}(\hat{H}_{\rm Gorkov} - \epsilon_-\sigma_3^{\rm cc})\Psi\right].$$

For clarity, explicit reference to the structure of the source term has been suspended. The latter can be restored when necessary.

---

[7] In the following it will be important to note that the transformation rules for supervectors and supermatrices differ from those of conventional vectors and matrices. In particular, if we define a pair of supervectors

$$\psi = \begin{pmatrix} S \\ \chi\end{pmatrix}, \qquad \bar{\psi} = \begin{pmatrix}\bar{S} & \bar{\chi}\end{pmatrix}$$

with commuting and anticommuting elements S, $\bar{S}$ and χ, $\bar{\chi}$ respectively, the supertransposition operation is defined according to

$$\psi^T = \begin{pmatrix} S & -\chi\end{pmatrix}, \qquad \bar{\psi}^T = \begin{pmatrix}\bar{S} \\ \bar{\chi}\end{pmatrix}.$$

Similarly, under a supertransposition, a supermatrix transforms as

$$F = \begin{pmatrix} S_1 & \chi_1 \\ \chi_2 & S_2\end{pmatrix}, \qquad F^T = \begin{pmatrix} S_1 & -\chi_2 \\ \chi_1 & S_2\end{pmatrix}, \qquad \text{i. e. } F \neq (F^T)^T.$$

2.1.2. *Impurity averaging*

To develop the low-energy theory of the disordered superconductor, the first step in the program is to implement the impurity average. The result will be to transform the free theory into an interacting theory. Separating the Gor'kov Hamiltonian into regular and stochastic parts as $\hat{H}_{\rm Gorkov} = \hat{H}^{(0)}_{\rm Gorkov} + W(\mathbf{r})\sigma_3^{\rm ph}$ and subjecting the generating function to an ensemble average over a Gaussian δ-correlated impurity distribution (8),

$$P(W)DW = \frac{e^{-\pi\nu\tau\int d\mathbf{r}\, W^2(\mathbf{r})}DW}{\int DW e^{-\pi\nu\tau\int d\mathbf{r}\, W^2(\mathbf{r})}}$$

one obtains

$$\langle \mathcal{Z}[0]\rangle_W = \int D[\bar\Psi,\Psi]\exp\left[\int d\mathbf{r}\ \left(i\bar\Psi(\hat{H}^{(0)}_{\rm Gorkov} - \epsilon_-\sigma_3^{\rm cc})\Psi \right.\right.$$
$$\left.\left. -\frac{1}{4\pi\nu\tau}(\bar\Psi\sigma_3^{\rm ph}\Psi)^2\right)\right].$$

In this form we can proceed in two ways: firstly, we could undertake a perturbative expansion in the interaction. Indeed, an appropriate rearrangement of the resulting series recovers the diagrammatic diffusion mode expansion. A second, and more profitable route, is to seek an appropriate mean-field decomposition of the interaction. Specifically, we are interested in identifying the diffusive modes discussed in chapter 1, *i.e.* two-particle channels arising from multiple scattering with momentum difference smaller than the inverse of the elastic mean free path, $\ell = v_F\tau$.

2.1.3. *Slow mode decoupling*

Isolating these modes is a standard, if technical, procedure [27] which is conveniently performed in Fourier space. Let us then focus on the quartic interaction generated by the impurity average:

$$\frac{1}{4\pi\nu\tau}\int d\mathbf{r}\ \left(\bar\Psi(\mathbf{r})\sigma_3^{\rm ph}\Psi(\mathbf{r})\right)^2.$$

From this term, we want to isolate within it the collective modes involving small momentum transfer, $|\mathbf{q}| < q_0 \sim 1/\ell$, which are to be decoupled by a Hubbard-Stratonovich transformation — these represent the soft modes identified in section 1.4. To achieve this, following Ref. [30], we present the interaction in the Fourier representation, viz.

$$\int d\mathbf{r}\ \left(\bar\Psi(\mathbf{r})\sigma_3^{\rm ph}\Psi(\mathbf{r})\right)^2$$
$$= \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \bar\Psi(\mathbf{k}_1)\sigma_3^{\rm ph}\Psi(\mathbf{k}_2)\ \bar\Psi(\mathbf{k}_3)\sigma_3^{\rm ph}\Psi(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3).$$

Now there are three independent ways of pairing two fast single-particle momenta to form a slow two-particle momentum $\mathbf{q}$:

	$\bar{\Psi}(\mathbf{k}_1)$	$\Psi(\mathbf{k}_2)$	$\bar{\Psi}(\mathbf{k}_3)$	$\Psi(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$
(a)	$\mathbf{k}$	$-\mathbf{k}+\mathbf{q}$	$\mathbf{k}'$	$-\mathbf{k}'-\mathbf{q}$
(b)	$\mathbf{k}$	$-\mathbf{k}'-\mathbf{q}$	$-\mathbf{k}+\mathbf{q}$	$\mathbf{k}'$
(c)	$\mathbf{k}$	$\mathbf{k}'$	$-\mathbf{k}'-\mathbf{q}$	$-\mathbf{k}+\mathbf{q}$

Term (a) can be decoupled trivially, producing no more than energy shifts that can be absorbed by a redefinition of the chemical potential. The other two terms can be rearranged in the following way. For term (b) we have

$$\begin{aligned}&\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \bar{\Psi}(\mathbf{k})\sigma_3^{\mathrm{ph}}\Psi(-\mathbf{k}'-\mathbf{q})\ \bar{\Psi}(-\mathbf{k}+\mathbf{q})\sigma_3^{\mathrm{ph}}\Psi(\mathbf{k}') \\ &= \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \bar{\Psi}(\mathbf{k})\sigma_3^{\mathrm{ph}}\Psi(-\mathbf{k}'-\mathbf{q})\ \Psi^T(\mathbf{k}')\sigma_3^{\mathrm{ph}}\bar{\Psi}^T(-\mathbf{k}+\mathbf{q}) \\ &= \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \bar{\Psi}(\mathbf{k})\sigma_3^{\mathrm{ph}}\Psi(-\mathbf{k}'-\mathbf{q})\left(-\bar{\Psi}(\mathbf{k}')\gamma^{-1}\sigma_2^{\mathrm{ph}}\right)\sigma_3^{\mathrm{ph}}\left(\gamma\sigma_2^{\mathrm{ph}}\Psi(-\mathbf{k}+\mathbf{q})\right) \\ &= \sum_{\mathbf{q}} \mathrm{str}\left[\left(\sum_{\mathbf{k}'}\Psi(-\mathbf{k}'-\mathbf{q})\otimes\bar{\Psi}(\mathbf{k}')\sigma_3^{\mathrm{ph}}\right)\left(\sum_{\mathbf{k}}\Psi(-\mathbf{k}+\mathbf{q})\otimes\bar{\Psi}(\mathbf{k})\sigma_3^{\mathrm{ph}}\right)\right].\end{aligned}$$

Here we have introduced the supertrace operation which acts on a supermatrix M according to $\mathrm{str}\ M = \mathrm{tr}\ M_{\mathrm{bb}} - \mathrm{tr}\ M_{\mathrm{ff}}$. Moreover, we have made use of the symmetry relations $\bar{\Psi}^T = \gamma\sigma_2^{\mathrm{ph}}\Psi$, and $\Psi^T = -\bar{\Psi}\gamma^{-1}\sigma_2^{\mathrm{ph}}$, which follow from Eq. (20). Finally, the term (c) is easily brought to the same form by using the cyclic invariance of the supertrace. Therefore, to assimilate the soft degrees of freedom, we may affect the replacement

$$\frac{1}{4\pi\nu\tau}\int d\mathbf{r}\ \left(\bar{\Psi}(\mathbf{r})\sigma_3^{\mathrm{ph}}\Psi(\mathbf{r})\right)^2 \simeq 2\times\frac{1}{4\pi\nu\tau}\sum_{|\mathbf{q}|<q_0}\mathrm{str}\left[\Gamma(-\mathbf{q})\Gamma(\mathbf{q})\right],$$

where the factor of 2 reflects the two channels of decoupling (b) and (c), and Γ is given by a sum of dyadic products of the fields Ψ and $\bar{\Psi}$

$$\Gamma(\mathbf{q}) = \sum_{\mathbf{k}}\Psi(-\mathbf{k}+\mathbf{q})\otimes\bar{\Psi}(\mathbf{k})\sigma_3^{\mathrm{ph}}.$$

(Note that, if the summation over $\mathbf{q}$ was unrestricted, the Hubbard-Stratonovich transformation would involve an overcounting by a factor of 2.)

With this definition, we can now implement a Hubbard-Stratonovich decoupling with the introduction of 8×8 supermatrix fields, Q,

$$\exp\left[-\frac{1}{2\pi\nu\tau}\sum_{\mathbf{q}}\mathrm{str}\left(\Gamma(\mathbf{q})\Gamma(-\mathbf{q})\right)\right]$$

$$= \int DQ \exp\left[\frac{1}{2\tau}\sum_{\mathbf{q}} \mathrm{str}\left(\frac{\pi\nu}{4}Q(\mathbf{q})Q(-\mathbf{q}) - Q(\mathbf{q})\Gamma(-\mathbf{q})\right)\right].$$

The symmetry properties of Q reflect those of the dyadic product $\Gamma(\mathbf{q})$. In particular, the symmetry relation

$$\begin{aligned}
\mathrm{str}\left[Q\Psi\otimes\bar{\Psi}\sigma_3^{\mathrm{ph}}\right] &= \mathrm{str}\left[\sigma_3^{\mathrm{ph}}\bar{\Psi}^T\otimes\Psi^T Q^T\right] \\
&= \mathrm{str}\left[\sigma_3^{\mathrm{ph}}(\gamma^{-1}\sigma_2^{\mathrm{ph}}\Psi)\otimes(-\bar{\Psi}\gamma\sigma_2^{\mathrm{ph}})Q^T\right] \\
&= \mathrm{str}\left[\sigma_2^{\mathrm{ph}}\gamma Q^T\gamma^{-1}\sigma_3^{\mathrm{ph}}\sigma_2^{\mathrm{ph}}\Psi\otimes\bar{\Psi}\right] \\
&= \mathrm{str}\left[\sigma_1^{\mathrm{ph}}\gamma Q^T\gamma^{-1}\sigma_1^{\mathrm{ph}}\Psi\otimes\bar{\Psi}\sigma_3^{\mathrm{ph}}\right],
\end{aligned}$$

is accounted for by subjecting the supermatrix Q to the linear condition

$$Q = \sigma_1^{\mathrm{ph}}\gamma\, Q^T\gamma^{-1}\sigma_1^{\mathrm{ph}}. \tag{22}$$

Finally, integrating out the fields Ψ, and $\bar{\Psi}$, and switching back to the coordinate representation, we obtain $\langle\mathcal{Z}[0]\rangle = \int DQ\ \exp\left[-S[Q]\right]$, where

$$S[Q] = -\int d\mathbf{r}\left[\frac{\pi\nu}{8\tau}\mathrm{str}\ Q^2 - \frac{1}{2}\ \mathrm{str}\ \ln\hat{\mathcal{G}}^{-1}\right]. \tag{23}$$

Here

$$\hat{\mathcal{G}}^{-1} = \hat{\zeta} + \sigma_3^{\mathrm{ph}}\hat{\Delta} - \epsilon_-\sigma_3^{\mathrm{cc}}\otimes\sigma_3^{\mathrm{ph}} + \frac{i}{2\tau}Q \tag{24}$$

represents the 'supermatrix' Green function with $\hat{\Delta} = |\Delta|\sigma_1^{\mathrm{ph}}e^{-i\varphi\sigma_3^{\mathrm{ph}}}$.

The domain of integration of the Hubbard-Stratonovich field Q is important. It is fixed by the requirement of convergence (in the boson-boson block), and this ultimately determines the structure of the saddle-point manifold of the σ-model. Historically, the first careful analysis of this issue is due to Weidenmüller, Verbaarschot and Zirnbauer [32] for the normal case. Later, Zirnbauer [17] provided a construction for each of the ten universality classes that emphasizes the algebraic aspects in ensuring convergence. In chapter 3 the integration manifold will be vital in our analysis of instanton saddle-points: we will specify the required contours there and refer to the literature for the details.

The problem of computing the disorder averaged Green function (and, if necessary, its higher moments) has been reduced to considering an effective field theory with the action $S[Q]$. Further progress is possible only within a saddle-point approximation.

2.1.4. *Saddle-point approximation and the σ-model*

The next step in deriving the low-energy theory is to explore the saddle-point structure of the effective action (23), and to classify and incorporate fluctuations by means of a gradient expansion. This is most straightforwardly achieved by implementing a two-step procedure devised in Ref. [10]. For in the dirty limit $\Delta \ll 1/\tau$, the scales set by the disorder and by the superconducting order parameter are well separated, so that one can perform two minimizations in sequence.

The strategy adopted in Ref. [10] is as follows: at first, one neglects the order parameter Δ and the deviation of the energy from the Fermi level, ϵ. By varying the resulting effective action, one finds the corresponding saddle-point manifold stabilized by the semi-classical parameter $\epsilon_F\tau \gg 1$. Then, fluctuations inside this manifold are considered; they couple to the order parameter and to the energy ϵ. The resulting low-energy effective action is varied once again inside the first (high-energy) saddle-point manifold. We will find that the corresponding low-energy saddle-point equation coincides with the Usadel equation (16) for the average quasi-classical Gor'kov Green function in the dirty limit.

In the absence of the order parameter, a variation of the action functional at the Fermi energy $S[Q]$ yields the saddle point equation:

$$Q(\mathbf{r}) = \frac{i}{\pi\nu}\mathcal{G}(\mathbf{r},\mathbf{r})$$

Taking the solution $Q_{\rm sp}$ to be spatially homogeneous, and setting $\int d\mathbf{p}/(2\pi)^d = \int \nu(\zeta)d\zeta \simeq \nu(0)\int d\zeta$, the saddle-point equation can be recast as

$$Q_{\rm sp} = \frac{i}{\pi}\int \frac{d\zeta}{\zeta - \epsilon_- \sigma_3^{\rm ph}\otimes\sigma_3^{\rm ph} + iQ_{\rm sp}/2\tau}, \tag{25}$$

where the positive infinitesimal 0^+ allows a distinction to be drawn between the physical and unphysical solutions. For $\epsilon_F\tau \gg 1$ the integral (25) may be evaluated in the pole approximation from which one obtains the diagonal matrix solution $Q = \mathrm{diag}(q_1, q_2, ...)$, with $q_i = \pm 1$. To choose the signs correctly, we note that the expression on the right-hand side of the saddle-point equation relates to the Green function of the disordered normal system evaluated in the self-consistent Born approximation. The disorder preserves the causal (i.e. retarded versus advanced) character of the Green function, and therefore the sign of q_i must coincide with the sign of the imaginary part of the energy. This singles out the particular solution

$$Q_{\rm sp} = \sigma_3^{\rm ph}\otimes\sigma_3^{\rm cc}.$$

As anticipated, however, this solution is not unique for $\epsilon \to 0$. Dividing out rotations that leave $\sigma_3^{\rm cc}\otimes\sigma_3^{\rm ph}$ invariant, the degeneracy of the manifold spanned by $Q = TQ_{\rm sp}T^{-1}$ is specified by the coset space $\mathrm{SU}(2,2|4)/\mathrm{SU}(2|2)\otimes\mathrm{SU}(2|2)$. The above form of $Q_{\rm sp}$ means that the manifold may also be defined by the non-linear condition $Q^2 = \mathbb{1}$.

Fluctuations transverse to this manifold are integrated out using the saddle-point parameter $\nu L^d/\tau \gg 1$. In the Gaussian approximation they do not couple to fluctuations on the saddle-point. Furthermore, the integration yields a factor of unity by supersymmetry [27] (for a more complete discussion see, e.g., Ref. [30]). With the saddle point approximation understood, it is straightforward to derive the σ-model action from (23) by inserting $Q(\mathbf{r}) = T(\mathbf{r})Q_{\rm sp}T^{-1}(\mathbf{r})$ into the expression (23) for $S[Q]$ and expanding in $\hat{\Delta}$ and ϵ, and up to second order in gradients of $Q(\mathbf{r})$, neglecting higher-order derivatives.

$$S[Q] = -\frac{\pi\nu}{8}\int d\mathbf{r}\ {\rm str}\left[D(\nabla Q)^2 - 4i(\hat{\Delta} + \epsilon_- \sigma_3^{\rm cc})\sigma_3^{\rm ph}Q\right] \ , \tag{26}$$

where $D = v_F^2\tau/d$ denotes the classical diffusion constant of the normal metal. The effect of a vector potential $\mathbf{A}$ is included by the replacement $\nabla \to \widetilde{\nabla} \equiv \nabla - ie\mathbf{A}[\sigma_3^{\rm ph},\ ]$ [27].

Let us emphasize the approximations used in the derivation of (26). Besides the quasi-classical ($\epsilon_F\tau \gg 1$) and saddle-point ($\nu L^d/\tau \gg 1$) parameters, one requires that all energies left are small compared to $1/\tau$, which allows us to truncate the expansion. Thus the action applies to $(Dq^2, \epsilon, \Delta) \ll 1/\tau$, where q is a wavevector characterizing the scale of variation of Q. This includes the usual dirty limit. We stress again that the completeness of the description provided by the action (26) within these approximations means that all physics at these energies should be contained.

This completes the derivation of the intermediate energy scale action. However, even on the soft manifold $Q^2(\mathbf{r}) = \mathbb{1}$, the majority of degrees of freedom are rendered massive by the order parameter and energy. To explore the structure of the low-energy action it is necessary to implement a further saddle-point analysis of (26) taking into account the influence of the superconducting order parameter.

2.1.5. *Low-energy saddle-point and soft modes*

To identify the low-energy saddle-point it is necessary to seek the optimal energy configuration of the supermatrix field Q for a non-vanishing order parameter $\hat{\Delta}$ and, in principle, a non-vanishing magnetic vector potential $\mathbf{A}$. We therefore require $S[Q]$ to be stationary with respect to variations of $Q(\mathbf{r})$ that preserve the non-linear constraint $Q^2(\mathbf{r}) = \mathbb{1}$. Following Ref. [30], such variations can be parametrized by transformations

$$\delta Q(\mathbf{r}) = \eta\left[X(\mathbf{r}), Q(\mathbf{r})\right],$$

where $X = -\sigma_1^{\rm ph}\gamma X^T\gamma^{-1}\sigma_1^{\rm ph}$ so as to preserve the symmetry (22). Subjecting the action to this variation, and linearizing in X, the stationarity condition $\delta S = 0$ translates to the equation of motion

$$D\widetilde{\nabla}\left(Q\widetilde{\nabla}Q\right) + i\left[Q, (\epsilon_-\sigma_3^{\rm cc} + \hat{\Delta})\sigma_3^{\rm ph}\right] = 0. \tag{27}$$

Associating $Q(\mathbf{r})$ with the average quasi-classical Gorkov Green function $g_0(\mathbf{r})$, the saddle-point equation is identified as the mean-field Usadel equation (16) derived in section 1.5. In hindsight the coincidence should not be surprising. At each stage of this calculation we have implemented approximations consistent with the quasi-classical scheme. With this understanding, we will tend to refer to the above as the Usadel equation.

Although the solution of this equation constrains many of the degrees of freedom to a single saddle-point, in the limit $\epsilon = 0$ several degrees of freedom remain massless for any value of the order parameter $\hat{\Delta}$. Specifically, the action functional $S[Q]$ is invariant under transformations

$$Q(\mathbf{r}) \mapsto TQ(\mathbf{r})T^{-1}, \quad \text{if} \quad T = \mathbb{1}^{\mathrm{ph}} \otimes t \tag{28}$$

with $t = \gamma(t^{-1})^T\gamma^{-1}$ constant in space. The latter condition means that t runs through an orthosymplectic Lie supergroup $\mathrm{OSp}(2|2)$. According to the classification scheme discussed in section 1.4, this defines the symmetry class CI. In presence of a magnetic field, the space of massless fluctuations is further diminished to the coset manifold $\mathrm{OSp}(2|2)/\mathrm{GL}(1|1)$ characterizing the symmetry class C. Not all of the classes are available to us in the present formulation. The classes designated D and DI require the introduction of spin degrees of freedom. This will be done in the next chapter, where we will encounter a realization of class D and the associated novel phase coherent phenomena.

This completes the formal construction of the low-energy statistical field theory of the weakly disordered superconductor. At the level of the mean-field of saddle-point, an application of this theory reproduces the results of the quasi-classical scheme. The role of fluctuations around the mean-field impacts most strongly on situations where low-energy quasi-particles are allowed to exist, e.g. quasi-particle states trapped around a vortex in the mixed phase [30], bulk superconductors driven into a gapless phase by a parallel magnetic field or magnetic impurities (see section 3), or hybrid superconductor/normal structures. To explore the impact of these novel mechanisms of quantum interference, in the following section we will explore the phenomenology of the magnetic impurity system. However, before doing so, let us first explore the mean-field structure of the action focusing on two simple examples: the bulk s-wave superconductor (and the restoration of the Anderson theorem), and the case of a quantum dot contacted to a superconducting terminal. Indeed, the latter solution will be needed in section 3.

2.2. DISORDERED BULK SUPERCONDUCTOR

In the absence of a magnetic field, taking the order parameter to be spatially homogeneous and specifying the gauge $\varphi = 0$ (i.e. $\hat{\Delta} = \sigma_1^{\mathrm{ph}}|\Delta|$), the saddle-point equation for Q can be solved straightforwardly. With the ansatz

$$Q_{\mathrm{sp}} = \sigma_3^{\mathrm{cc}} \otimes \sigma_3^{\mathrm{ph}} \cosh\hat{\theta} - i\sigma_2^{\mathrm{ph}} \sinh\hat{\theta} \tag{29}$$

where the matrix $\hat{\theta}$ is diagonal in the superspace with elements $\hat{\theta} = \mathrm{diag}(\theta_\mathrm{b}, \theta_\mathrm{f})$, the saddle-point or Usadel equation assumes the form

$$D\nabla^2\theta + 2\left(|\Delta|\cosh\hat{\theta} - \epsilon_-\sinh\hat{\theta}\right) = 0 \tag{30}$$

Taking $\hat{\theta}$ to be homogeneous with $\theta_\mathrm{b} = \theta_\mathrm{f}$, we obtain the BCS solution $\theta = \theta_\mathrm{s}$ (19).

Having obtained the quasi-classical Green's function we can impose the self-consistency condition $\Delta = -(\lambda/\nu)\langle\psi_\downarrow\psi_\uparrow\rangle$ as usual. We obtain the gap equation

$$|\Delta| = i\lambda\pi T\sum_{\epsilon_n}\sinh\theta|_{\epsilon_-=i\epsilon_n} \tag{31}$$

where the summation is taken over fermionic Matsubara frequencies $\epsilon_n = \pi T(2n+1)$. Similarly, from the saddle-point solution, we obtain the quasi-particle DoS

$$\begin{aligned}\nu(\epsilon) &= \frac{1}{\pi}\mathrm{tr}\ \mathrm{Im}\ \hat{G}^-(\epsilon) = -\frac{1}{4\pi}\mathrm{tr}\ \mathrm{Im}\ \hat{\mathcal{G}}\sigma_3^\mathrm{ph}\otimes\sigma_3^\mathrm{cc} \\ &= \frac{\nu_\mathrm{n}}{4}\mathrm{Re}\ \mathrm{str}\left[\sigma_3^\mathrm{bf}\otimes\sigma_3^\mathrm{cc}\otimes\sigma_3^\mathrm{ph}\ Q\right] = 2\nu_\mathrm{n}\mathrm{Re}\ \cos\theta(\epsilon)\ ,\end{aligned}$$

just as in the usual quasi-classical theory.

We finish this first example with an important technical comment. In the present formalism the above result follows from a saddle-point approximation. Yet normally any quantity calculated in this way is weighted by a factor $e^{-S[Q_\mathrm{sp}]}$. To complete the correspondence with the usual quasi-classical theory, we note that the saddle-point Q_sp should be chosen proportional to unity in the boson-fermion space. Through the definition of the supertrace, this ensures that $S[Q_\mathrm{sp}] = 0$. In the same way, any fluctuation corrections to the saddle-point action vanish by supersymmetry [27].[8] Saddle-point configurations that are not 'supersymmetric' in this sense can be important and we will discuss such a case in the next chapter.

2.3. HYBRID SN-STRUCTURES

With the Usadel equation in hand, one can proceed (once the correct boundary conditions are known) to find solutions in more complex geometries, that describe hybrid superconductor-normal systems [10]. In chapter 3 we will need the mean-field result for a geometry that cannot in fact be described by the Usadel equation as it stands. This is the case of a quantum dot contacted to a superconductor.

2.3.1. *Quantum dot contacted to a superconductor*

The case of a normal quantum dot coupled to superconducting lead through a contact of arbitrary transparency (see Fig. 5) presents us with a dilemma. The

[8] Of course, fluctuations are important in the calculation of non-supersymmetric source terms used to extract physical quantities from the action.

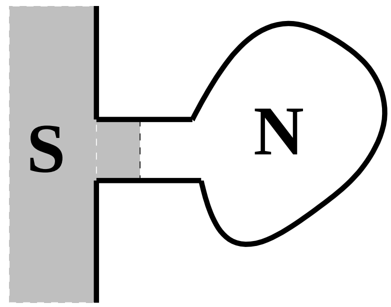

Figure 5. Metallic quantum dot coupled to a superconducting lead.

lead has N propagating modes. The quantum dot is a small metallic region with $D/L^2 \gg N\delta$.[9] The energy scale that determines the influence of the contact on the properties of the dot is the inverse of the time taken for an electron in the dot to feel the contact. This defines the generalized Thouless energy [33], and for the quantum dot, this scale is set by $N\delta$ (modulo factors relating to the transparency of the lead). In a large dot with $D/L^2 \ll \delta$ the diffusive motion of the electrons would set this scale.

A naive expectation is that this problem should involve the solution of the Usadel equation as before, with the right boundary conditions. The above considerations show this not to be the case. With D/L^2 the largest energy scale in the problem, gradients of Q are frozen out of the action. One must explicitly include the coupling to the leads from the outset, as the saddle point will be determined by the competition between the energy ϵ and this coupling (of order $N\delta$) in the action. D/L^2 will appear nowhere. Put simply, the gradient expansion is not the true low-energy action in such a confined geometry.

Unfortunately, a fully microscopic derivation of the correct form of the zero-dimensional (that is, containing no spatial gradients) action is laborious [27]. We can get to the answer more directly by using the general principle that the zero-dimensional limit of the action describes the appropriate random matrix model, or equivalently, that the quantum dot system in the limit $D/L^2 \gg \delta$ may be modeled by random matrix theory with matrices of size $M \to \infty$, as described in section 1.4. The random matrix model for the dot is simply a Gor'kov Hamiltonian (3) with $\Delta = 0$ — the dot is normal — and $\hat{H}$ given by an appropriate random Hamiltonian with mean level spacing δ from the orthogonal symmetry class. The non-trivial element is the coupling to the leads. The standard approach [34] is to

[9] This includes the case of a ballistic chaotic quantum dot, provided the ergodic time (the time required for an electron to explore the available phase space) is much longer than the dwell time of the electrons in the dot.

write the lead-dot coupling as

$$\hat{H}_{\mathrm{LD}} = \sum_{j,\alpha} \int \frac{dk}{2\pi} (W_{\alpha j} (|\alpha, p\rangle\langle j, k, p| - |\alpha, h\rangle\langle j, k, h|) + \mathrm{h.c.}) \ . \tag{32}$$

In this expression $|\alpha, n\rangle$ with $n = p, h$ denotes a basis of the random matrix model for the dot, and $|j, k, n\rangle$ is the obvious basis for the $j = 1 \ldots N$ propagating modes of the lead. Though this coupling is formally the same as a tunneling Hamiltonian it is capable of describing contacts of arbitrary transparency with proper interpretation of the couplings $W_{\alpha j}$. It is possible to show that the dot can be described by the 'effective Hamiltonian',[10]

$$\hat{H}_{\mathrm{eff}} \equiv \hat{H}\sigma_3^{\mathrm{ph}} - i\pi\nu W W^\dagger \hat{g}_{\mathrm{bcs}}(\epsilon) \ ,$$

where g_{bcs} is defined in Eq. (18). It is this structure that is needed in the derivation of the zero-dimensional σ-model. By expanding only in ϵ in the '$\mathrm{str}\ln$' form of the action (23) one arrives at

$$S[Q] = \frac{i\pi\epsilon_-}{2\delta} \mathrm{str}\left[\sigma_3^{\mathrm{cc}} \otimes \sigma_3^{\mathrm{ph}} Q\right] - \frac{1}{2}\sum_j \mathrm{str}\left[\ln(1 + \alpha_j Q_{\mathrm{bcs}} Q)\right], \tag{33}$$

where Q_{bcs} is used to denote the bulk BCS saddle-point found in the previous section. In the above we have taken $WW^\dagger$ to be the $M \times M$ diagonal matrix $\mathrm{diag}\{\alpha_1, \ldots, \alpha_N, 0, \ldots, 0\}$. (33) is the proper form of the σ-model for a quantum dot with superconducting leads. It was first used by [35] in their investigation of the class C spectral statistics of such a device. Since we are typically interested in energies of the order of the level spacing, the order parameter may be taken to infinity so that $Q_{\mathrm{bcs}} = \sigma_1^{\mathrm{ph}}$. We will specialize at this stage to the case of N perfectly ballistic contacts, so that all $\alpha_j = 1$.

As before, to obtain a mean-field expression for the DoS it is necessary to minimize the action with respect to variations in Q. Doing so, one obtains the saddle-point equation

$$-\frac{i\pi\epsilon_-}{2\delta}[Q, \sigma_3^{\mathrm{cc}} \otimes \sigma_3^{\mathrm{ph}}] + \frac{N}{2}[Q, (1 + Q_{\mathrm{bcs}} Q)^{-1} Q_{\mathrm{bcs}}] = 0$$

Applying the ansatz that the saddle-point solution is contained within the diagonal parameterization (29), the saddle-point equation takes the form

$$-\frac{\pi\epsilon_-}{\delta}\sinh\hat{\theta} + \frac{N}{2}\frac{\cosh\hat{\theta}}{1 + i\sinh\hat{\theta}} = 0. \tag{34}$$

[10] This has a well-defined meaning only within the context of a scattering approach [34]. For an informal derivation, write down the BdG equations (2) for the whole system and eliminate states from outside the dot.

We can straightforwardly determine that there is a 'minigap' $E_{\rm gap}$ in the DoS by setting $\cosh\theta_{\rm sp}$ to be imaginary. Thus $\sinh\theta_{\rm s} \equiv -ib$ for real b and (34) gives

$$\epsilon(b) = \frac{N\delta}{2\pi}\frac{1}{b}\sqrt{\frac{b-1}{b+1}}\,.$$

The extremum of this function gives the largest energy corresponding to a real value of b. This occurs at $b = (1+\sqrt{5})/2 = 1+\gamma$, where γ is the golden mean, and yields $E_{\rm gap} = (N\delta/2\pi)\gamma^{5/2} \approx 0.048N\delta$. With a bit more effort, one can expand in the vicinity of $E_{\rm gap}$ to obtain

$$\nu(\epsilon) \simeq \begin{array}{ll} 0 & \epsilon < E_{\rm gap}, \\ \frac{1}{\pi L^d}\sqrt{\frac{\epsilon - E_{\rm gap}}{\Delta_g^3}} & \epsilon < E_{\rm gap}, \end{array} \qquad (35)$$

where $\Delta_g \approx 0.068N^{1/3}\delta$.

Finally, we note that, in the opposite case of α_j small, one can expand the logarithm in α_j. In the first order the action is just the same as for a BCS superconductor with gap $(\delta/\pi)\sum_j \alpha_j$. The formation of the minigap is a highly non-trivial effect. Indeed, in Ref. [33], the integrity of the gap is proposed as a signature of irregular or chaotic dynamics inside the dot. A dot with integrable dynamics appears to possess only a 'soft' gap in the DoS, with the DoS going to zero at zero energy. It is no surprise that 'diffusive' SN structures, where the gradient action and Usadel equation are the appropriate description, also display a minigap. For a modern theoretical review of minigap structures in superconductor/normal compounds, see Ref. [36].

This completes our study of the mean-field spectral properties of the hybrid superconducting/normal system. In principle, these results could have been recovered without resort to the field theoretic scheme. To address the importance of mesoscopic fluctuations on the coherence properties of the superconducting system, we now turn to a bulk system which exhibits low-energy quasi-particle excitations. Here we will require the full machinery of the non-linear σ-model.

3. Superconductors with magnetic impurities: instantons and sub-gap states

3.1. INTRODUCTION

In section 1.2 we discussed Anderson's observation that the thermodynamic properties of an s-wave superconductor in the dirty limit are independent of the amount of normal (non-magnetic) impurities added to the system. In the argument the time-reversal symmetry of the single-particle Hamiltonian plays a prominent role: pairing occurs between degenerate time-reversed eigenstates. When time-reversal symmetry is broken we expect pairing to be disrupted and superconductivity suppressed. This can be achieved by applying a magnetic field or by adding magnetic impurities. The effect is described by the classic theory of Abrikosov and Gor'kov [13] (AG), who considered the magnetic impurity case, though the description has a high degree of universality [37].

It is easy to see the importance of time-reversal symmetry from the Gor'kov Hamiltonian

$$\hat{H}_{\text{Gorkov}} = \begin{pmatrix} \hat{H} & \Delta\sigma_2^{\text{sp}} \\ \Delta^*\sigma_2^{\text{sp}} & -\hat{H}_0^T \end{pmatrix}_{\text{ph}} . \tag{36}$$

This differs from Eq. (3) through the introduction of the spin space (with Pauli matrices denoted σ_i^{sp}). The Pauli matrix σ_2^{sp} in the off-diagonal particle-hole block reflects singlet pairing. We introduce scattering by normal and magnetic impurities through the simple model

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} - \epsilon_F + W(\mathbf{r}) + J\mathbf{S}(\mathbf{r}) \cdot \sigma^{\text{sp}} . \tag{37}$$

In addition to the weak potential impurity distribution $W(\mathbf{r})$, the particles experience a quenched random magnetic impurity distribution $J\mathbf{S}(\mathbf{r})$ where J represents the exchange coupling. The inclusion of $J\mathbf{S}(\mathbf{r})$ evidently prevents the simple diagonalization of (36) in terms of the single-particle eigen-energies as before.

AG solved the model defined by Eq. (36) together with the self-consistent equation for the order parameter (4) in the self-consistent Born approximation. Their results are expressed in terms of the spin-flip scattering rate $1/\tau_s$ through the natural dimensionless parameter

$$\zeta \equiv \frac{1}{\tau_s|\Delta|} . \tag{38}$$

The relation between $1/\tau_s$ and $J\mathbf{S}(\mathbf{r})$ will be given shortly. In section 1.3 we explained how a time-reversal symmetry breaking perturbation leads to the suppression of superconductivity (in the present model $1/\tau_\varphi = 2/\tau_s$). This certainly has the flavour of a mesoscopic effect: it depends on the loss of phase rigidity in the single-particle wavefunctions as the time-reversal symmetry is broken.[11] It is, however, of a 'mean-field' character. In this chapter we will see that a complete description of the DoS within the model defined by Eq. (37) necessitates the inclusion of non-perturbative effects as well as the novel channels of quantum phase coherence discussed in the introduction.

[11] This notion of phase rigidity can be made precise. In Ref. [38] the 'order parameter' $\rho \equiv \langle|\int d\mathbf{r}\phi_\alpha^2|\rangle$ is calculated for the crossover from the orthogonal ($\rho = 1$) to the unitary ($\rho = 0$) symmetry classes.

3.1.1. *Density of states*

We saw that AG's formula (11) followed from general considerations and it is indeed universal [37, 4]. Quantities such as the quasi-particle DoS are more model dependent. In the present model AG found that, remarkably, the suppression of the gap in the DoS is more rapid than that of the superconducting order parameter (Fig. 6). They found a narrow 'gapless' superconducting phase in which the quasi-particle energy gap is destroyed while the superconducting order parameter remains non-zero. This prediction was soon confirmed experimentally.

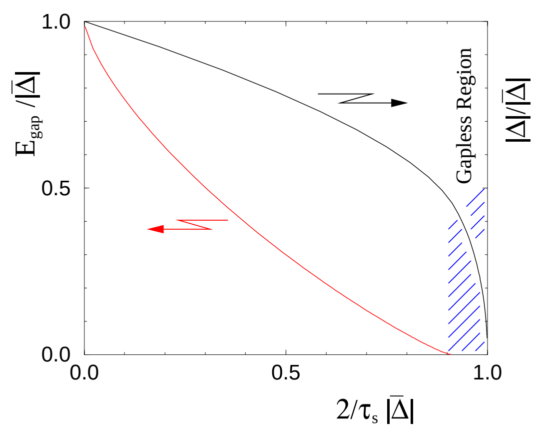

Figure 6. Variation of the energy gap E_{gap} and the self-consistent order parameter $|\Delta|$ as a function of (normalized) scattering rate $2/\tau_{\mathrm{s}}|\bar{\Delta}|$. $|\bar{\Delta}|$ is the order parameter at $1/\tau_s = 0$.

This immediately presents two questions:

1. According to AG, the gap is maintained up to a critical concentration of magnetic impurities (at $T = 0$, 91% of the critical concentration at which superconductivity is destroyed). Yet, being unprotected by the Anderson theorem, it seems likely that the gap structure predicted by the mean-field theory is untenable and must be subject to non-perturbative corrections. What is the structure of the resulting 'sub-gap' states?
2. The gapless superconducting phase has quasi-particle states all the way down to zero energy. These low energy states should be strongly affected by channels of quantum interference discussed in section 1.4. Where does the gapless system fit into this classification and what are the consequences for the spectral and transport properties?

Once identified, the answer to the second question can be straightforwardly inferred from existing studies of the relevant universality class. Here we will be more concerned with answering the first question.

Sub-gap states in the magnetic impurity system have been discussed before. Strong magnetic impurities [39–41] evidently lie outside the Born approximation used by AG. In particular it was shown that, in the unitarity limit, a single magnetic impurity leads to the local suppression of the order parameter and creates a bound sub-gap quasi-particle state [39]. For a finite impurity concentration, these intra-gap states broaden into a band [40] merging smoothly with the continuum bulk states.

We will argue that there is a mesoscopic view of this problem which is more universal. Sub-gap states are those which are anomalously lacking in phase rigidity in the presence of a time-reversal symmetry breaking perturbation. This could be either an extrinsic or intrinsic effect. By intrinsic we mean that this is simply what happens to some proportion of states of this random Hamiltonian when we switch on such a perturbation. Alternatively, one can conceive of an extrinsic mechanism: The AG theory shows the gap to follow the relation

$$E_{\rm gap}(\tau_s) = |\Delta| \left(1 - \zeta^{2/3}\right)^{3/2} \tag{39}$$

showing an onset of the gapless region at $\zeta = 1$ (note $\hbar = 1$ throughout). Even for weak disorder, however, it is apparent that optimal fluctuations of the random potential must generate sub-gap states in the interval $0 < \zeta < 1$, thus providing non-perturbative corrections to the self-consistent Born approximation used by AG. A fluctuation of the random potential which leads to an effective Born scattering rate $1/\tau_s'$ in excess of $1/\tau_s$ over a range set by the superconducting coherence length,

$$\xi = \left(\frac{D}{|\Delta|}\right)^{1/2}, \tag{40}$$

induces quasi-particle states down to energies $E_{\rm gap}(\tau_s')$.[12] These sub-gap states are localized, being bound to the region where the scattering rate is large, see Fig. 7. We will return to this picture later.

The situation bears comparison with band tail states in semi-conductors. In this instance, rare or optimal configurations of the random impurity potential generate bound states, known as Lifshitz tail states [43], which extend below the band edge. The correspondence is, however, somewhat superficial: band tail states in semi-conductors are typically associated with smoothly varying, nodeless wave-functions. By contrast, the tail states below the superconducting gap involve the superposition of states around the Fermi level. As such, one expects these states to be rapidly oscillating on the scale of the Fermi wavelength λ_F, but modulated by an envelope which is localized on the scale of the coherence length ξ. This difference is not incidental. Firstly, unlike the semi-conductor, one expects the energy dependence of the density of states in the tail region below the mean-field gap edge to be ‘universal’, independent of the nature of the weak impurity distribution

[12] Similar arguments have been made by Balatsky and Trugman [42].

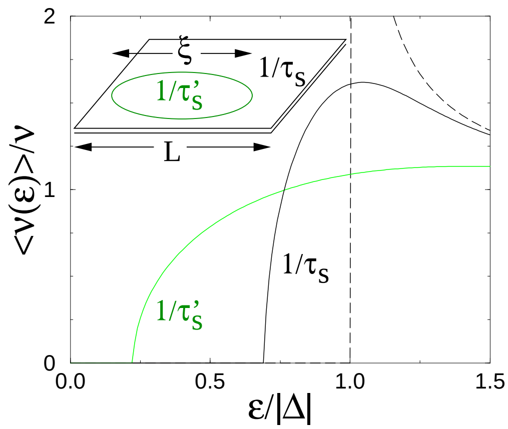

Figure 7. Mechanism of extrinsic sub-gap state formation.

but dependent only on the pair-breaking parameter ζ. Secondly, as we will see, one can not expect a straightforward extension of existing theories [43, 44] of the Lifshitz tails to describe the profile of tail states in the superconductor.

3.1.2. *Outline*

In this chapter, following Refs. [45], we will first show how to extend the statistical field theory described in chapter 2 to incorporate scattering by magnetic impurities. As anticipated in the previous chapter, a saddle-point approximation recovers the mean-field theory of AG. We discuss the soft-modes of the action that exist in the gapless phase and determine the consequences of these new channels of interference. In section 3.4, with the field theory in hand, we turn to problem of the sub-gap states. We find that these are described by instantons of the field theory; we identify the profile of the instanton with the envelope modulating the quasi-classical sub-gap states. A careful analysis allows us to evaluate the sub-gap density of states with exponential accuracy. In section 3.5 we examine the zero dimensional limit and prove a recent universality conjecture [46]. We next discuss the universality of the $d > 0$ problem in the context of other realizations of gapless superconductivity.

3.2. FIELD THEORY OF THE MAGNETIC IMPURITY PROBLEM

Incorporating the additional structure of (36) into the field theoretic description obtained in the previous chapter is straightforward. As before one starts from the generating functional

$$\mathcal{Z}[J] = \int D(\bar{\psi}, \psi) e^{\int d\mathbf{r}\, \left(i\bar{\psi}(\hat{H}_{\rm Gorkov} - \epsilon_-)\psi + \bar{\psi}J + \bar{J}\psi\right)}, \tag{41}$$

where $\epsilon_- \equiv \epsilon - i0$ and the supervector fields have the internal structure $\bar{\psi} = (\bar{\psi}_\uparrow\ \bar{\psi}_\downarrow\ \psi_\uparrow\ \psi_\downarrow)$, $\psi^T = (\psi_\uparrow\ \psi_\downarrow\ \bar{\psi}_\uparrow\ \bar{\psi}_\downarrow)$. As in chapter 2 we will only be concerned with the average of a single Green's function.

3.2.1. *σ-model action*

For clarity it is desirable to remove the $\sigma_2^{\rm sp}$ from the off-diagonal terms in Eq. (36). To do this, we perform the rotation $\psi \mapsto \psi' = U\psi$, $\bar{\psi} \mapsto \bar{\psi}' = \bar{\psi}U^\dagger$ with

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i\sigma_2^{\rm sp} \end{pmatrix}_{\rm ph} ,$$

after which the Gor'kov Hamiltonian takes the form

$$\hat{H}_{\rm Gorkov} = \left(\frac{\hat{\mathbf{p}}^2}{2m} + W(\mathbf{r}) - \epsilon_F\right) \otimes \sigma_3^{\rm ph} + J\mathbf{S}(\mathbf{r}) \cdot \sigma^{\rm sp} + |\Delta|\sigma_2^{\rm ph} .$$

Since, in the following, the global phase can be chosen arbitrarily, the order parameter can be chosen to be real. The unusual phase coherence properties of the superconducting system rely on the particle/hole or charge conjugation symmetry

$$\hat{H}_{\rm Gorkov} = -\sigma_2^{\rm ph} \otimes \sigma_2^{\rm sp} \hat{H}_{\rm Gorkov}^T \sigma_2^{\rm sp} \otimes \sigma_2^{\rm ph}. \tag{42}$$

As before, one can include all channels of interference by further doubling the field space as in chapter 2. Rather than present all the intermediate steps, we give only the symmetry relation on Q, the Hubbard-Stratonovich field introduced to decouple the average over W. In this case

$$Q = \sigma_1^{\rm ph} \otimes \sigma_2^{\rm sp} \gamma Q^T \gamma^{-1} \sigma_1^{\rm ph} \otimes \sigma_2^{\rm sp} , \tag{43}$$

where now, in contrast to Eq. 21, we have defined

$$\gamma = \mathbb{1}^{\rm ph} \otimes \begin{pmatrix} i\sigma_2^{\rm cc} & \\ & \sigma_1^{\rm cc} \end{pmatrix}_{\rm bf} .$$

We will see presently that, when there are quasi-particle states at low energy in the present system, their localization properties are radically different to those of systems in the previous chapter. It is through this new γ that the distinction enters the present formalism.

Turning to the magnetic impurity scattering due to the $J\mathbf{S}(\mathbf{r}) \cdot \sigma^{\rm sp}$ term, we use the Gaussian model specified by zero mean and variance

$$\left\langle JS_\alpha(\mathbf{r}) JS_\beta(\mathbf{r}')\right\rangle_S = \frac{1}{6\pi\nu\tau_s}\delta^d(\mathbf{r}-\mathbf{r}')\delta_{\alpha\beta} , \tag{44}$$

where $1/\tau_s$ is the spin flip scattering rate introduced earlier.[13] Averaging yields the term in the Ψ field action

$$\left\langle \exp\left[i\int d\mathbf{r}\Psi J\mathbf{S}(\mathbf{r})\cdot\sigma^{\mathrm{sp}}\bar{\Psi}\right]\right\rangle_{J\mathbf{S}} = \exp\left[-\frac{1}{12\pi\nu\tau_s}\int d\mathbf{r}(\bar{\Psi}\sigma^{\mathrm{sp}}\Psi)^2\right] . \quad (45)$$

The interaction generated by the magnetic impurity averaging can be treated [27] by performing all possible pairings and making use of the saddle-point approximation $Q(\mathbf{r}) = 2\langle\Psi(\mathbf{r})\otimes\bar{\Psi}(\mathbf{r})\sigma_3^{\mathrm{ph}}\rangle_\Psi/\pi\nu$. This leads to the replacement

$$\frac{1}{12\pi\nu\tau_s}\int d\mathbf{r}\ \left(\bar{\Psi}\sigma^{\mathrm{sp}}\Psi\right)^2 \mapsto \frac{\pi\nu}{24\tau_s}\int d\mathbf{r}\ \mathrm{str}\ \left(Q\sigma_3^{\mathrm{ph}}\otimes\sigma^{\mathrm{sp}}\right)^2 .$$

Such an approximation, which neglects pairings at non-coincident points is allowed by the strong inequality $(\ell/\xi)^d \ll 1$. In addition we discard the contraction $\langle\bar{\Psi}\sigma^{\mathrm{sp}}\Psi\rangle_\Psi$. The term generated by this procedure could in any case be decoupled by a slow bosonic field $\mathbf{S}(\mathbf{r})$ which would immediately be set to zero for the singlet saddle-points that will be the basis of this section.

Gaussian in the fields Ψ and $\bar{\Psi}$, the functional integration can be performed explicitly after which one obtains $\langle\mathcal{Z}[0]\rangle_{V,S} = \int DQ\exp(-S[Q])$ where

$$S[Q] = -\int d\mathbf{r}\left[\frac{\pi\nu}{8\tau}\mathrm{str}\ Q^2 - \frac{1}{2}\mathrm{str}\ \ln\left(\sigma_3^{\mathrm{ph}}(\hat{\mathcal{H}}_0 - \epsilon_-\sigma_3^{\mathrm{cc}}) + \frac{i}{2\tau}Q\right)\right.$$
$$\left. -\frac{\pi\nu}{24\tau_s}\mathrm{str}\ (Q\sigma_3^{\mathrm{ph}}\otimes\sigma^{\mathrm{sp}})^2\right] .$$

From this point, the σ-model follows precisely as before

$$S[Q] = -\frac{\pi\nu}{8}\int d\mathbf{r}\ \mathrm{str}\ \left[D(\nabla Q)^2 - 4i\left(\epsilon_-\sigma_3^{\mathrm{cc}} + |\Delta|\sigma_2^{\mathrm{ph}}\right)\sigma_3^{\mathrm{ph}}Q\right.$$
$$\left. -\frac{1}{3\tau_s}\left(Q\sigma_3^{\mathrm{ph}}\otimes\sigma^{\mathrm{sp}}\right)^2\right] . \quad (46)$$

The saddle point manifold is given by $Q = TQ_{\mathrm{sp}}T^{-1}$, with $Q_{\mathrm{sp}} = \sigma_3^{\mathrm{ph}}\otimes\sigma_3^{\mathrm{cc}}$ and T chosen to be consistent with (43). The quasi-particle DoS is obtained from the functional integral

$$\langle\nu(\epsilon,\mathbf{r})\rangle_{V,S} = \frac{\nu}{4}\mathrm{Re}\ \left\langle\mathrm{str}\left(\sigma_3^{\mathrm{bf}}\otimes\sigma_3^{\mathrm{ph}}\otimes\sigma_3^{\mathrm{cc}}Q(\mathbf{r})\right)\right\rangle_Q . \quad (47)$$

The numerical factor leads to a DoS of 4ν for the system as $|\epsilon|\to\infty$. This is because both the particle-hole structure of the original Bogoliubov Hamiltonian

[13] Following AG, we take the quenched distribution of magnetic impurities to be 'classical' and non-interacting throughout — indeed, otherwise our method would not apply in its present formulation. For practical purposes, this entails the consideration of structures where both the Kondo temperature [47] and, more significantly, the RKKY induced spin glass temperature [48] are smaller than the relevant energy scales of the superconductor.

and the spin each cause a doubling of the DoS. With the appropriate extension of the σ-model in hand, we should now check that the mean-field description of AG is recovered at the saddle-point level, as anticipated.

3.2.2. *AG Mean-Field Theory*

Variation of the σ-model action with respect to fluctuations of Q obtains the Usadel equation

$$D\nabla\left(Q\nabla Q\right)+i\left[Q,\epsilon_{-}\sigma_3^{\rm cc}\otimes\sigma_3^{\rm ph}+i|\Delta|\sigma_1^{\rm ph}\right] + \frac{1}{6\tau_s}\left[Q,\sigma_3^{\rm ph}\otimes\sigma^{\rm sp}Q\sigma_3^{\rm ph}\otimes\sigma^{\rm sp}\right]=0. \qquad (48)$$

With the ansatz

$$Q_{\rm sp}=\left[\sigma_3^{\rm cc}\otimes\sigma_3^{\rm ph}\cosh\hat\theta+i\sigma_1^{\rm ph}\sinh\hat\theta\right]\otimes \mathbb{1}^{\rm sp}, \qquad (49)$$

where the elements $\hat\theta={\rm diag}(\theta_1,i\theta)_{\rm bf}$ are diagonal in the superspace, the saddle-point equation decouples into boson-boson and fermion-fermion sectors, and takes the form

$$\nabla^2_{\mathbf{r}/\xi}\hat\theta+2i\left(\cosh\hat\theta-\frac{\epsilon}{|\Delta|}\sinh\hat\theta\right)-\zeta\sinh(2\hat\theta)=0\ . \qquad (50)$$

As explained in section 2.2 we take $\hat\theta=\theta_1\mathbb{1}^{\rm bf}$ and spatially constant to recover the results of the usual Usadel theory [24] for this problem. Together with the self-consistency equation (31) we have[14]

$$\begin{aligned} 0&=\epsilon\sinh\theta_1-|\Delta|\cosh\theta_1-\frac{i}{\tau_s}\sinh(2\theta_1),\\ |\Delta|&=-i\pi\lambda\int d\epsilon\ \sinh\theta_1(\epsilon). \end{aligned} \qquad (51)$$

The saddle-point equations (51) can be solved self-consistently following the procedure outlined, for example, in Ref. [37]. Setting $\epsilon=\widetilde\epsilon-(1/2\tau_s)\cosh\theta_1$ and $|\Delta|=|\widetilde\Delta|+(1/2\tau_s)\sinh\theta_1$, the saddle-point equation for each energy ϵ takes the form $\widetilde\epsilon\sinh\theta_1=|\widetilde\Delta|\cosh\theta_1=0$. Setting $\widetilde\upsilon\equiv\widetilde\epsilon/|\widetilde\Delta|$ and recalling the definition $\zeta=1/\tau_s|\Delta|$, one obtains

$$\upsilon\equiv\frac{\epsilon}{|\Delta|}=\widetilde\upsilon\left(1-\zeta\frac{1}{\sqrt{1-\widetilde\upsilon^2}}\right).$$

To reiterate, the latter equation should be regarded as a self-consistent solution for $\widetilde\upsilon$ from which one can obtain $\theta=\arcsin(1/\sqrt{1-\widetilde\upsilon^2})$. The corresponding

[14] Here we work at zero temperature.

self-consistent equation for the gap parameter then takes the form

$$|\Delta| = -\pi\lambda \int d\epsilon \, \frac{1}{\sqrt{1-\widetilde{\upsilon}^2}}.$$

Although there is no simple closed analytic expression for the solution of the mean-field equation, much is known about its form. In particular, the system exhibits a transition at $\zeta = 1$ from a gapped to a gapless phase. In the gapped phase, i.e. for $\zeta < 1$, the gap edge is fixed by the solution $\widetilde{\upsilon}_{\rm gap} = \left(1-\zeta^{2/3}\right)^{1/2}$, from which one obtains $E_{\rm gap} = \Delta\left(1-\zeta^{2/3}\right)^{3/2}$. A numerical solution for the AG DoS for various values of the dimensionless parameter ζ is shown in Fig. 8.

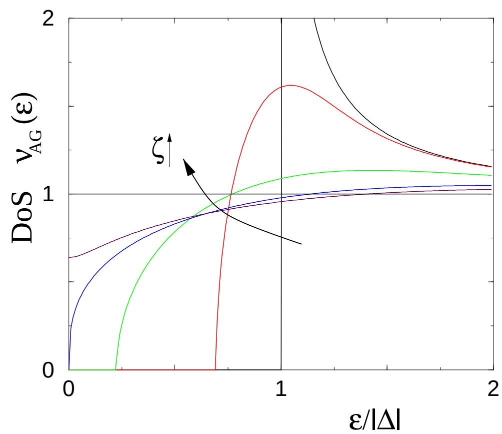

Figure 8. Average DoS as obtained from the Abrikosov-Gor'kov mean-field theory for $\zeta = 0$, 0.1, 0.5, 1, 1.3 and ∞. Note that for $\zeta > 1$, the system enters the gapless phase with the DoS at $\epsilon = 0$ non-vanishing.

Turning to the self-consistent equation, at $T = 0$, the gap equation can be written in the form,

$$1 = -\lambda \int_0^{\omega_D} \frac{d\epsilon}{|\Delta|} \frac{1}{(1+\widetilde{\upsilon}^2)^{1/2}}$$

Changing the integration variable from ϵ to $\widetilde{\upsilon}$, the gap equation assumes the form,

$$1 = -\lambda \int_{\widetilde{\upsilon}_l}^{\omega_D/\Delta} d\widetilde{\upsilon} \left[1 - \zeta \frac{1}{(1+\widetilde{\upsilon}^2)^{3/2}}\right] \frac{1}{(1+\widetilde{\upsilon}^2)^{1/2}}$$

where the lower limit is defined by

$$\widetilde{\upsilon}_l = \begin{array}{ll} 0 & \zeta \le 1, \\ (\zeta^2-1)^{1/2} & \zeta > 1. \end{array}$$

The integration can be performed analytically and obtains the solution,

$$\ln\left(\frac{\Delta}{\bar{\Delta}}\right) = \begin{array}{ll} -\frac{\pi}{4}\zeta & \zeta \leq 1 \\ -\mathrm{arccosh}\zeta - \frac{1}{2}\left(\zeta \arcsin(1/\zeta) - (1-1/\zeta^2)^{1/2}\right) & \zeta > 1 \end{array}$$

where $\bar{\Delta}$ is the order parameter for $\zeta = 0$. For small ζ, Δ decreases linearly,

$$\Delta - \bar{\Delta} = -\frac{\pi}{4\tau_s}.$$

The onset of the gapless region occurs when $\zeta = 1$. At this point, $\Delta = \bar{\Delta}e^{-\pi/4}$, from which one obtains $1/\tau_s = \bar{\Delta}e^{-\pi/4}$. Using the fact that superconductivity completely disappears when $1/\tau_s = \bar{\Delta}/2$, one finds that the gapless region arises at 91% of the critical impurity concentration [13]. The variation of the self-consistent order parameter and quasi-particle energy with $2/\tau_s|\bar{\Delta}|$ is shown in shown in Fig. 6.

3.3. PHASE COHERENCE EFFECTS IN THE GAPLESS PHASE

Having determined the homogeneous solution of the mean-field equation we now turn attention to influence of fluctuations. As in the time-reversal invariant bulk s-wave superconductor, in the gapped region of the phase diagram, fluctuations are rendered massive by the energy ϵ. Here the fluctuations serve only to provide a small renormalization of the mean-field DoS above the gap. However, in the gapless phase, quasi-particle states persist to zero energy. In this limit, some fluctuations become soft.

More precisely, in the limit $\epsilon \to 0$, the mean-field solution (49) to the saddle-point equation (48) is not unique: here the saddle-point equation admits an entire manifold of homogeneous solutions parameterized by the transformations $Q = TQ_{\mathrm{sp}}T^{-1}$ where $T = \mathbb{1}_{\mathrm{ph}} \otimes \mathbb{1}_{\mathrm{sp}} \otimes t$ and $t = \gamma(t^{-1})^T\gamma^{-1}$: soft fluctuations of the fields, which are controlled by a non-linear σ-model defined on the group manifold $T \in \mathrm{OSp}(2|2)/\mathrm{GL}(1|1)$. This corresponds to symmetry class D in the Altland-Zirnbauer classification scheme discussed in section 1.4. This is no surprise as this class corresponds to Gor'kov Hamiltonians with broken time-reversal and spin rotation symmetry. The massless fluctuations control the low-energy, long-range properties of the gapless system giving rise to unusual localization and spectral properties.

Taking into account slow spatial fluctuations of the fields

$$Q(\mathbf{r}) = T(\mathbf{r})Q_{\mathrm{sp}}T^{-1}(\mathbf{r}) = (\sinh\theta_1\sigma_1^{\mathrm{ph}} + \cosh\theta_1\,\sigma_3^{\mathrm{ph}}Q_s(\mathbf{r})) \otimes \mathbb{1}^{\mathrm{sp}},$$

where $Q_s(\mathbf{r}) = T(\mathbf{r})\sigma_3^{\mathrm{cc}}T^{-1}(\mathbf{r})$, one obtains the soft mode non-linear σ-model action

$$S_{Q_s} = -\frac{\pi\nu}{8}\int d\mathbf{r}\,\mathrm{str}\left[D_s(\nabla Q_s)^2 - 4i\epsilon_s\sigma_3^{\mathrm{cc}}Q_s\right], \tag{52}$$

where $D_s = D\cos^2\theta_1$ denotes the effective diffusion constant, and $\epsilon_s = \epsilon\cosh\theta_1$ represents the (potentially complex) energy source. The mean-field solution therefore brings about an energy dependent renormalization of the coupling constants in the non-linear σ-model action. We emphasize that, these constants aside, the form of the soft-mode action (by which we mean the structure of the saddle-point manifold) — and therefore low-energy behaviour — is entirely determined by the symmetry of the original random Hamiltonian. In the present case, the Q-matrix relation (43) follows from the symmetry (42) of the Hamiltonian. This point of view is most elegantly put by Zirnbauer [17]. Fortunately extensive analysis of class D through the action (52) exists in the literature [49–52], and we can just quote the results. These fall into two categories

Thermal transport: Since neither charge nor spin are conserved in the original Hamiltonian, novel effects with be present only in *thermal* transport by quasiparticles. A standard perturbative RG analysis of (52) yields for 2D the flow

$$\frac{dg(L)}{d(\ln L/\ell)} = \frac{1}{\pi^2} \ ,$$

g now gives the thermal conductivity, and we see that transport is metallic. Indeed, the differing behaviours of classes C and D can be attributed to the following observation [50]. Friedan's [53] general result on the one-loop β-function of a σ-model in 2D shows that the flow is determined by the curvature of the field space. The different γ matrices used in this chapter and the last indicate that moving from class C to D involves switching the manifolds of the boson-boson and fermion-fermion sectors, which reverses the curvature (in the appropriate superspace generalization) and leads to *anti-localization* in the class D case.

Spectral properties: One of the common features of the superconductor universality classes [16] is non-stationary behaviour of the DoS due to the distinguished position of $\epsilon = 0$. The correction to the mean-field (AG in our case) DoS can be calculated in perturbation theory for $|\epsilon_s| \gg E_c = |D_s|/L^2$. In the 2D case we have

$$\begin{aligned}\nu(\epsilon) &= \mathrm{Re}\,\nu\cosh\theta_1\left(1 + \frac{1}{\pi\nu}\int\frac{d^2q}{(2\pi)^2}\frac{1}{D_sq^2 - 2i\epsilon_s}\right)\\ &= \mathrm{Re}\,\nu\cosh\theta_1\left(1 + \frac{1}{8\pi^2 D_s\cosh^2\theta_1}\ln\left(1 + \left(\frac{D_s}{2\epsilon_s\ell^2}\right)^2\right)\right). \quad (53)\end{aligned}$$

For $|\epsilon_s| < E_c$, the action is dominated by the zero spatial mode, which must be integrated exactly. This is the universal limit, and the result should coincide with the average DoS of a random matrix ensemble with the class D symmetry (see section 1.4). The result is [16, 50]

$$\nu(\epsilon) = \nu(E_c)\left[1 + \frac{\sin(2\pi\epsilon/\delta)}{2\pi\epsilon/\delta}\right]$$

where $\delta = 1/\nu(E_c)L^2$ represents the average energy level spacing at energy E_c. This accounts for the RG scaling from the microscopic scale to the Thouless scale E_c suggested by (53).

Thus, according to the class D field theory, the DoS obtained from the AG mean-field should be modulated by a logarithmic divergence at low energies which is cut on the scale E_c. Below E_c, the modulation should be universal corresponding to the random matrix result. In particular, as $\epsilon/\delta \to 0$, the DoS should exhibit a jump by a factor of two.

This completes the formal description of the bulk superconducting phase. The solution of the AG mean-field equation provides an adequate description of the bulk extended states. New channels of quantum interference are described by soft modes in the gapless phase with dramatic consequences. This answers the second of the two questions posed in the introduction. Now we move onto the first: the sub-gap structure in the gapped phase.

3.4. INSTANTONS AND SUB-GAP STATES

Although the reduction and eventual destruction of the quasi-particle energy gap predicted by the AG mean-field theory can be reasonably justified on purely physical grounds, the integrity of the gap of the range $0 < \zeta < 1$ is less credible. Once time-reversal symmetry is broken and the protection of Anderson's theorem is lost, there remains no reason why a sharp gap should persist. Add to this the observation that the spin scattering rate must be subject to spatial fluctuations from the average value $1/\tau_s$, and one concludes that corrections to the DoS predicted by the AG theory must lead to the appearance of sub-gap states analogous to "band tails" in a disordered semiconductor [43, 44].

This analogy is of course not new [42, 54] nor, as far as practical calculation in the present formulation is concerned, is it particularly deep. This is because all averages have already been taken, so we can not look for an optimal fluctuation of some potential, as in the classic approaches to the study of band tail states in disordered semi-conductors [44]. However, these studies hint at how one can proceed.

Band tail states in semi-conductors can be studied within the same functional integral formulation. In particular, the generating function of the single-particle Green function of a normal disordered conductor can be presented in the form of a supersymmetric field integral

$$\mathcal{Z}[0] = \int D(\Psi, \bar{\Psi}) \exp\left[i \int d\mathbf{r} \bar{\Psi} \left(\epsilon_+ - \frac{\hat{\mathbf{p}}^2}{2m} - W(\mathbf{r})\right) \Psi\right],$$

where, once again, the random impurity distribution is drawn from a Gaussian δ-correlated white-noise impurity potential. The optimal fluctuation method involves minimizing the action with respect to fluctuations in the fields Ψ and potential W. This involves seeking inhomogeneous solutions of the non-linear Schrödinger equation

$$\left(\epsilon - \frac{\hat{\mathbf{p}}^2}{2m} - W(\mathbf{r})\right)\Psi = 0,$$

where the corresponding optimal potential is determined self-consistently by the relation $W(\mathbf{r}) = -|\Psi(\mathbf{r})|^2/2\pi\nu\tau$. In the supersymmetric formulation, band tail states are identified with 'supersymmetry broken' inhomogeneous solutions of the saddle-point equation (see Cardy [55] and Affleck [56]). Indeed, the anticipated exponential suppression of the DoS necessitates a breaking of supersymmetry to support a finite action. Here the phrase "supersymmetry breaking" is potentially misleading. We use it only to refer to *field configurations*, ubiquitous in the problems under discussion here, that do not respect the parity between Bose and Fermi degrees of freedom. However, any such configuration is just one member of a degenerate manifold differing by supersymmetric transformations. The latter maintain the invariance of the generating functional $\mathcal{Z}[0]$ under global supersymmetric transformations.

What does this tell us about the identification of optimal fluctuations and sub-gap states in the superconductor? Following the analysis above, one might guess that sub-gap states are associated with inhomogeneous configurations of the Ψ field action. However, we anticipate that optimal solutions corresponding to sub-gap states are localized on a length scale in excess of the superconducting coherence length. In the dirty limit, $\xi \gg \ell \gg \lambda_F$, this implies that the localized sub-gap states are quasi-classical in nature. Their existence on the level of the Ψ field action will be buried in the fast λ_F oscillations of the wavefunction. To reveal the sub-gap states, we must first remove the fast short length scale fluctuations of the quasi-classical Green function and look for an equation of motion for the slowly varying envelope of the wavefunction. But this is just the program of the usual quasi-classical scheme.

The term "sub-gap states" is a little misleading in this context. Band tails are bound states of some rare potential that sit by themselves below the bulk of the spectrum. Each rare configuration that make the gap soft in the present case will give rise to many states beneath the AG gap. Thus the term "gap fluctuation", used in Ref. [46] to describe the zero dimensional SN system, may be more appropriate.

As well as being quasi-classical in nature, the existence of sub-gap states is not affected by working in the dirty limit. As such, their existence must be accommodated in the non-linear σ-model functional (46) since the validity of this description relied only on the quasi-classical parameter $\epsilon_F\tau \gg 1$ and the dirty limit assumption. To identify sub-gap states in the present formalism, we should therefore investigate inhomogeneous solutions of the low-energy saddle-point equation in Q — the Usadel equation [25]. Such a solution should be thought of as defining an envelope for the quasi-classical sub-gap states.

Therefore, let us revisit the mean-field equation and look for inhomogeneous solutions at energies $\epsilon < E_{\rm gap}$. To focus our discussion, let us begin by restricting attention to the quasi one-dimensional geometry. To stay firmly within the diffusive regime, we therefore impose the requirement that the system size L be much smaller than the localization length of the normal system $L_{\rm loc.} \simeq \nu d_\perp^2 D$, where $d_\perp$ is the wire diameter. Later, in section 3.4.3, we will generalize our discussion to encompass systems of higher dimension. Furthermore, since, over the interval $0 < \zeta < 1$, the quasi-particle energy gap varies more rapidly than the superconducting order parameter, we will neglect self-consistency of the order parameter. Taking self-consistency into account will not alter our qualitative findings, and will only weakly affect the quantitative results.

3.4.1. *Instantons in the Quasi One-dimensional Geometry*

To investigate inhomogeneous solutions of the mean-field equation (50) it is convenient to recast the equation in terms of its first integral

$$(\partial_{x/\xi}\hat{\theta})^2 + V(\hat{\theta}) = \text{const}, \tag{54}$$

where

$$V(\hat{\theta}) = 4i\left(\sinh\hat{\theta} - \frac{\epsilon}{\Delta}\cosh\hat{\theta}\right) - \zeta\cosh 2\hat{\theta}$$

denotes the complex potential. Let us denote by $\theta_{\rm AG}$ the values of θ_1 and $i\theta$ at the conventional saddle point, and focus on an energy ϵ below the gap predicted by the AG theory. Here $\text{Im }\theta_{\rm AG} = \pi/2$ such that the mean-field DoS $\nu_{\rm AG}(\epsilon) = 4\nu\text{Re }\cosh\theta_{\rm AG}$ vanishes. The corresponding value of $\text{Re }\theta_{\rm AG}$ depends sensitively on the energy, with $\text{Re }\theta_{\rm AG} = 0$ for $\epsilon = 0$.

Considering the boson-boson sector only, if we require that $\theta_1(x \to \pm\infty) = \theta_{\rm AG}$, what kind of inhomogeneous solution is possible? The values of θ_1 at which $\partial_x\theta_1 = 0$ can be identified by considering the complex (dimensionless) potential function $V(\theta_1)$ from which we can determine the endpoints of the 'motion' in the complex plane, just as one would use a real potential normally. By inspection one may see that, on the line $\text{Im }\theta_1 = \pi/2$, the potential is purely real. This is not the only contour where $\text{Im }V = 0$, but, by considering forces, it is not hard to see that either $\text{Im }\theta_1 = \pi/2$ always during the motion, or θ_1 follows a trajectory with an endpoint at $\text{Im }\theta_1 < 0$. For reasons outlined below, we will discount this latter possibility. The former case amounts to considering "bounce" trajectories in the *real* potential $V(i\pi/2 + \phi) = V_{\rm r}(\phi)$ where

$$V_{\rm r}(\phi) \equiv -4\left(\cosh\phi - \frac{\epsilon}{|\Delta|}\sinh\phi\right) + \zeta\cosh 2\phi. \tag{55}$$

A typical potential is shown in Fig. 9.

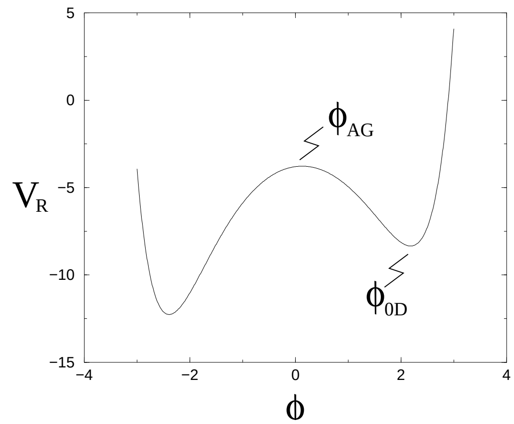

Figure 9. Potential $V_{\rm r}(\phi) = V(i\pi/2 + \phi)$ for $\epsilon/|\Delta| = 0.1$ and $\zeta = 0.2$. The AG saddle point corresponds to the central maximum. The saddle point marked $\phi_{0{\rm d}}$ is used in the analysis of the zero-dimensional problem (section 3.5).

Now integration over the angles $\hat{\theta}$ is constrained to certain contours [27]. Is the bounce solution accessible to both? As usual, the contour of integration over the boson-boson field θ_1 includes the entire real axis, while for the fermion-fermion field, $i\theta$ runs along the imaginary axis from 0 to $i\pi$. With a smooth deformation of the integration contours, the AG saddle-point is accessible to both the angles $\hat{\theta}$ [10]. By contrast, the bounce solution *and* the AG solution can be reached simultaneously by a smooth deformation of the integration contour *only* for the boson-boson field θ_1 (see Fig. 10). The bounce solution is therefore associated with a *breaking of supersymmetry* at the level of the saddle point.

Thus we have identified an inhomogeneous saddle-point configuration for which the supersymmetry is broken: θ_1 executes a bounce whilst $i\theta$ remains at the mean-field value $\theta_{\rm AG}$. The symmetry broken solution then incurs the (finite) real action

$$S = 4\pi\nu L_W (D|\Delta|)^{1/2} S_\phi(\epsilon/|\Delta|, \zeta)$$

where, defining ϕ' as the endpoint of the motion,

$$S_\phi \equiv \int_{\phi_{\rm AG}}^{\phi'} d\phi \sqrt{V_{\rm r}(\phi_{\rm AG}) - V_{\rm r}(\phi)}. \tag{56}$$

Now, as mentioned above, there exists a second possibility for a bounce solution in which one moves away from $\theta_{\rm AG}$ parallel to the imaginary axes. Indeed, such a solution would seem to be a natural candidate for the fermion-fermion field $i\theta$. However, since the endpoint for this trajectory lies at ${\rm Re}\ \theta < 0$ outside the integration domain which runs from 0 to π, this would seem to be excluded.

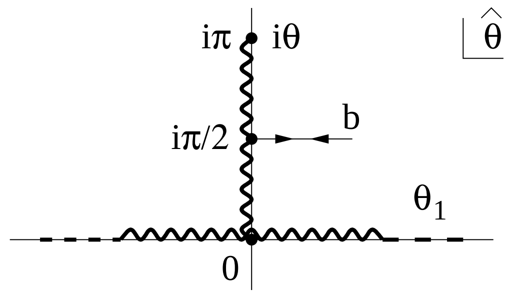

Figure 10. Integration contours for boson-boson and fermion-fermion fields in the complex $\hat{\theta}$ plane. The bounce solution for $\epsilon = 0$ (labelled as 'b') is shown schematically.

As ϵ approaches $E_{\rm gap}$ from below, the potential (55) becomes more shallow, with the maximum merging with one of the minima when we reach the gap. Near the edge, up to an irrelevant constant, an expansion of the potential in powers of $(\phi - \phi_{\rm AG})$ leads to the cubic form

$$V_{\rm r}[\phi] \simeq -\alpha \left(\frac{E_{\rm gap} - \epsilon}{|\Delta|}\right)^{1/2} (\phi - \phi_{\rm AG})^2 + \beta(\phi - \phi_{\rm AG})^3 \tag{57}$$

where the dimensionless coefficients are specified by

$$\alpha = 6\sqrt{\frac{2}{3}} \left(\frac{E_{\rm gap}}{|\Delta|}\right)^{1/6}, \qquad \beta = 2\left(\frac{\zeta E_{\rm gap}}{|\Delta|}\right)^{1/3}. \tag{58}$$

Note that, making use of Eq. (39), both of these coefficients depend solely on the dimensionless parameter ζ. From this expansion, one can obtain an analytic solution for S_ϕ. To leading order in $(E_{\rm gap} - \epsilon)/|\Delta|$ one finds

$$S_\phi = \frac{4}{15} \frac{\alpha^{5/2}}{\beta^2} \left(\frac{E_{\rm gap} - \epsilon}{|\Delta|}\right)^{5/4}. \tag{59}$$

Note that the action vanishes exactly at the gap. For completeness we give the explicit form of the bounce solution

$$\phi(x) - \phi_{\rm AG} = \frac{\alpha}{\beta} \frac{1}{\cosh^2(x/2r_0)} ,$$

where the extent of the instanton is set by

$$r_0(\epsilon) = \frac{\xi}{\alpha^{1/2}} \left(\frac{|\Delta|}{E_{\rm gap} - \epsilon}\right)^{1/4}. \tag{60}$$

Indeed the size of the instanton is easily understood from the quadratic "stiffness" term in Eq. (57). Thus one finds that, while the overall scale is set by the superconducting coherence length ξ, the size of the localized region diverges both as

ϵ approaches $E_{\rm gap}$ and, noting that $\alpha \sim (1-\zeta^{2/3})^{1/4}$, as one approaches the gapless phase $\zeta \to 1$.

This completes the analysis of the saddle-point solution together with the corresponding action. However, as this level we are presented with two problems:

1. The contribution of a second saddle point would seem to spoil the normalization condition $\langle \mathcal{Z}[0]\rangle_{V,S} = 1$, which should be preserved within the saddle point approximation;
2. Confined to the line $\mathrm{Im}\,\theta = \pi/2$, when substituted into the DoS source (47), the bounce configuration does not appear to generate states!

The resolution of both problems lies in the nature of the fluctuations around the symmetry broken mean-field solution. These field fluctuations can be separated into "radial" and "angular" contributions. The former involve fluctuations of the diagonal elements $\hat{\theta}$, while the latter describe rotations including those Grassmann transformations which mix the bf sector. Both classes of fluctuations play a crucial role.

3.4.2. *Fluctuations*

Let us outline qualitatively the influence of the fluctuations around the mean-field. As usual, associated with radial fluctuations around the bounce, there exists a zero mode (due to translational invariance of the solution), and a negative energy mode. The latter, which necessitates a $\pi/2$ rotation of the corresponding integration contour to follow the line of steepest descent (c.f. Ref. [57]), has two effects: firstly it ensures that the non-perturbative contributions to the local DoS are non-vanishing, and secondly, that they are positive. Turning to the angular fluctuations, the breaking of supersymmetry is accompanied by the appearance of a Grassmann zero mode separated by a gap from higher excitations which restores the global supersymmetry (c.f. spin symmetry breaking in a ferromagnet of finite extent). The zero mode ensures that the symmetry broken inhomogeneous saddle-point configurations respect the normalization condition $\langle \mathcal{Z}[0]\rangle_{V,S} = 1$.

A careful analysis of the fluctuations to formally check all these features is contained in Ref. [45]. The result is that, taking into account Gaussian fluctuations and zero modes, one obtains the non-perturbative, one instanton contribution to the sub-gap DoS:

$$\frac{\langle \nu(\epsilon)\rangle_{V,S}}{4\nu} \sim (-i|K|) \int dx\; i(\sinh\phi(x) - \sinh\phi_{\rm AG})\, |\varphi_0^-(x)|^2 \sqrt{\frac{LS_\phi}{\xi}} \times \exp\left[-4\pi\nu L_{\rm w}\sqrt{D|\Delta|}S_\phi\right] , \tag{61}$$

where the factor $\sqrt{LS_\phi/\xi}$ represents the Jacobian associated with the introduction of the collective coordinate [57], $-i|K|$ is the overall factor arising from the non-zero modes, and the Grassmann zero mode wavefunction φ_0^- is normalized such that $\int dx|\varphi_0^-|^2 = 1$. Eq. (61) is the main result of this section. Note the non-perturbative nature of the result, both in the coupling constant g^{-1} of the σ-model, and the (dimensionless) spin scattering rate ζ.

3.4.3. *Sub-gap States in Dimensions* $1 < d < 6$

The calculation above was tailored to the consideration of the quasi one-dimensional geometry. The generalization to higher dimensions follows straightforwardly. In particular, it is necessary to seek inhomogeneous solutions of the saddle-point equation (54) where the gradient operator must be interpreted as the higher dimensional generalization. Generally, this equation must be solved numerically. However, for energies ϵ in the vicinity of the gap $E_{\rm gap}$, an analytic expression for the energy scaling can be obtained for $d < 6$.[15]

Using the approximation to $V_{\rm r}[\phi]$ (57) valid when $(E_{\rm gap} - \epsilon)/|\Delta| \ll 1$, the exponential dependence of the sub-gap DoS can be deduced in higher dimension. In this limit, dimensional analysis of the cubic equation of motion yields the scaling form

$$\phi(\mathbf{r}) - \phi_{\rm AG}(\epsilon) = \frac{\alpha}{\beta} f(\mathbf{r}/r_0) \ ,$$

where r_0 is the characteristic length defined by Eq. (60). When substituted back into the action, one finds that the DoS depends exponentially on the parameter $4\pi g(\xi/L)^{d-2} S_\phi$

$$S_\phi = a_d \, \zeta^{-2/3} (1 - \zeta^{2/3})^{-(2+d)/8} \left(\frac{E_{\rm gap} - \epsilon}{|\Delta|} \right)^{(6-d)/4} . \qquad (62)$$

Here $g = \nu D L^{d-2}$ denotes the bare dimensionless conductance of the normal system, and a_d is a numerical constant ($a_1 = 8 \sqrt[4]{24}/5$). In particular, the exponent depends linearly on the energy separation from the gap in two dimensions.

Having completed this calculation, it is interesting to explore the connection of the results presented above to related problems in the literature. The resulting expression for the DoS (62) is non-perturbative in the σ-model coupling $1/g$, which measures the strength of non-magnetic disorder. We note that other non-perturbative results in disordered systems have been obtained by related instanton calculations. As well as the investigation of tail states in semi-conductors [55], a

[15] The importance of $d = 6$ is the following: the intuition from the quasi one-dimensional problem is that since the potential $V_{\rm r}$ becomes more shallow as $\epsilon \to E_{\rm gap}^-$ the action for the instanton is well approximated by retaining only those terms due to the cubic potential (57). This is correct in $d < 6$ because the instanton found using the truncated potential have an action that vanishes as $\epsilon \to E_{\rm gap}^-$, and it is clear that the 'remainder' of the action evaluated on this instanton vanishes even more rapidly. For $d > 6$ this procedure fails and no truncation of the potential is possible. Effectively the potential is $V_{\rm r} \sim \zeta \exp(2|\phi|)$ and no instanton solutions exist. Thus for $d > 6$ the mean-field gap is hard.

supersymmetric field theory was developed by Affleck [56] (see also Refs. [58]) to investigate tail states in the lowest Landau level. There it was shown that tail states correspond to instanton configurations of the Ψ-*field action* (c.f. Ref. [55]). It is also interesting to compare the present scheme with the study of 'anomalously localized states' [59] (see also, Ref. [60]). There one finds that long-time current relaxation in a disordered wire is also associated with instanton configurations of the σ-model action. Finally, a Lifshitz argument has been applied on the level of the Usadel equation in the study of gap fluctuations due to inhomogeneities of the BCS interaction [64].

3.5. ZERO DIMENSIONAL PROBLEMS AND UNIVERSALITY

In the previous section we considered the instanton contribution to the sub-gap DoS in the infinite system. For completeness let us now consider the zero dimensional case that obtains when r_0, the size of the instanton, exceeds the system size L, which will happen when ϵ approaches close enough to $E_{\rm gap}$ from below in any finite system. In this limit one can clearly not fit an instanton inside the system. Leaving aside the practical relevance of this situation, theoretical motivation is provided by a recent paper [54] that explored this regime using a random matrix analysis.

However, before turning to the consideration of the zero-dimensional limit of the present problem let us first try to draw some intuition from a system that turns out to be closely related. In section 2.3.1 we discussed the properties of a normal quantum dot contacted to a superconducting terminal (Fig. 5). There we saw that near the gap edge the DoS of the dot takes the singular form

$$\nu(\epsilon > E_{\rm gap}) \simeq \frac{1}{\pi L^d}\sqrt{\frac{\epsilon - E_{\rm gap}}{\Delta_{\rm g}^3}}\ , \tag{63}$$

However, the location of the gap edge relies on a mean-field analysis of the coupled system. In Ref.[46] Vavilov *et al.* have argued that optimal fluctuations of the impurity potential give rise to gap fluctuations. The hypothesis introduced in Ref. [46] is that the spectral statistics near a gap edge are universal. This allows a random matrix theory analysis of gap fluctuations and leads to the following expression for the sub-gap DoS,

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\frac{2}{3}\left(\frac{E_{\rm gap}-\epsilon}{\Delta_{\rm g}}\right)^{3/2}\right]. \tag{64}$$

Now the AG mean-field solution for a superconducting quantum dot with magnetic impurities also predicts the existence of a square root edge (see Eq. (35)). Then, when recast in the form of Eq. (63), it is pertinent to ask whether the expression for the sub-gap DoS coincides with Eq. (64) in the zero dimensional limit. This is the situation addressed in Ref.[54].

In the following we will show that the universal result (64) is explicitly recovered by the present theory. Moreover, in doing so, we will expose the origin of the universal structure reported in Ref. [46] and describe its implications for universality of the $d > 0$ problem.

3.5.1. *Superconducting dot with magnetic impurities*

Let us therefore consider explicitly the action of a superconducting grain in the presence of a weak magnetic impurity potential. When $r_0 \gg L$ only the zero spatial mode contributes significantly to the action (46). In this limit, the action assumes the zero dimensional form

$$S[Q] = \frac{i\pi}{2\delta}\mathrm{str}\left[\left(\epsilon_-\sigma_3^{\mathrm{cc}} + |\Delta|\sigma_2^{\mathrm{ph}}\right)\sigma_3^{\mathrm{ph}}Q\right] + \frac{\pi}{24\tau_s\delta}\mathrm{str}\left(Q\sigma_3^{\mathrm{ph}}\otimes\sigma^{\mathrm{sp}}\right)^2, \quad (65)$$

where, as usual, δ denotes the single-particle level-spacing. As in the higher dimensional problem, a variation of the action with respect to Q obtains a mean-field equation, now without spatial variation. Parameterizing the saddle-point equation as in section 3.2.2, we obtain the zero-dimensional Usadel or AG mean-field equation (50)

$$2i\left(\cosh\hat{\theta} - \frac{\epsilon}{|\Delta|}\sinh\hat{\theta}\right) - \zeta\sinh(2\hat{\theta}) = 0.$$

From this equation, we can identify the usual AG solution which in turn recovers the AG phenomenology.

The inclusion of bounce configurations in the previous calculations was based upon the observation that, although the contribution they make is exponentially small, they are the least action configurations on the part of the contour that gives a finite sub-gap DoS. In the zero-dimensional case we are spared having to think about the problem in function space. The action is proportional to the potential of Fig. 9. The correct contour thus passes through the maximum of the potential (minimum of the action) corresponding to the usual AG saddle point, and turns away from the real ϕ axes (i.e. the line $\mathrm{Im}\ \theta_1 = \pi/2$) at the minimum of the potential (marked in Fig. 9 as $\phi_{0\mathrm{d}}$). This part of the contour, parallel to the imaginary axes, gives a contribution to the DoS, and the second saddle point is in fact a *maximum* on this portion by analyticity. Following the same arguments as in section 3.4.1, this solution is inaccessible to the fermionic contour. Thus we find that the sub-gap DoS in the zero dimensional case is given by

$$\frac{\nu(\epsilon < E_{\mathrm{gap}})}{\nu} \sim \exp\left[-\pi\frac{|\Delta|}{\delta}(V_{\mathrm{r}}(\phi_{\mathrm{AG}}) - V_{\mathrm{r}}(\phi_{0\mathrm{d}}))\right] . \quad (66)$$

As before analytic results may be obtained near the gap edge using the cubic potential (57). We find

$$\phi_{0\mathrm{d}}(\epsilon) \sim \phi_{\mathrm{AG}}(\epsilon) + \frac{2\alpha}{3\beta}\sqrt{\frac{E_{\mathrm{gap}} - \epsilon}{|\Delta|}},$$

which leads to the sub-gap DoS near $E_{\rm gap}$ scaling as

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\frac{4\pi}{27}\frac{|\Delta|}{\delta}\frac{\alpha^3}{\beta^2}\left(\frac{E_{\rm gap}-\epsilon}{|\Delta|}\right)^{3/2}\right] \ .$$

We note that the general result for the energy dependence of the exponent written down in section 3.4.3 for dimensions $d \geq 1$ applies also for $d = 0$.

To establish contact with the universal result given in Eq. (63) it is helpful to recast the result in a modified form. To do this we note that, in the vicinity of the mean-field gap edge, the DoS can be expanded as [37]

$$\frac{\nu_{\rm AG}(\epsilon > E_{\rm gap})}{4\nu} \simeq \sqrt{\frac{2}{3}}\,\zeta^{-2/3}(1-\zeta^{2/3})^{-1/4}\left(\frac{\epsilon - E_{\rm gap}}{|\Delta|}\right)^{1/2} \ .$$

Then, if we define

$$\Delta_{\rm g}^{-3/2} \equiv \frac{4\pi}{\delta}\sqrt{\frac{2}{3|\Delta|}}\zeta^{-2/3}(1-\zeta^{2/3})^{-1/4} \ ,$$

the mean-field DoS can be brought to the form of Eq. (63), and the sub-gap DoS takes the universal form

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\frac{4}{3}\left(\frac{E_{\rm gap}-\epsilon}{\Delta_{\rm g}}\right)^{3/2}\right] \ . \tag{67}$$

Taking into account the particular convention for the definition of the DoS[16], the sub-gap DoS obtained above coincides with the universal expression shown in Eq. (64).

Away from the mean-field gap edge, we can in principle obtain an exact expression for the exponential dependence of the sub-gap DoS by solving the saddle-point equation for $V_{\rm r}(\phi_{\rm AG})$ and $V_{\rm r}(\phi_{\rm 0d})$ explicitly. However, such a program can not be performed analytically. However, the asymptotic dependence of the DoS tail far from the mean-field gap can be obtained by developing an asymptotic expansion in ζ. In doing so, we can establish contact with the results of Ref. [54]. Taking $\zeta \ll 1$ and keeping terms at order ζ^{-1} and ζ^0 in the potential $V_{\rm r}$ one finds that $\phi_{\rm AG}$ coincides with the BCS solution at leading order and $\phi_{\rm 0d}$ moves to the larger value

$$\phi_{\rm 0d}(\epsilon) \sim \ln\left(2\frac{|\Delta|-\epsilon}{|\Delta|\zeta}\right) .$$

[16] We note that the factor of 2 discrepancy between the result here and that presented in Ref. [46] can be straightforwardly accommodated into a redefinition of $\Delta_g^{-3/2}$, or, equivalently, a rescaling of the mean-field DoS. For reasons outlined in the text, we believe that the convention adopted here is the consistent one.

Evaluating $V_{\rm r}(\phi_{\rm AG})$ and $V_{\rm r}(\phi_{\rm 0d})$ and substituting in Eq. (66) gives the result

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\frac{2\pi\tau_s}{\delta}(|\Delta| - \epsilon)^2 + \frac{4\pi}{\delta}(|\Delta|^2 - \epsilon^2)^{1/2}\right] .$$

In particular, we note that the two terms in this expansion have the same functional form as those obtained in Ref. [54], the coefficients differ. The reason for this difference is unclear.

The rescaling of the DoS above and the appearance of the universal form suggests that we should revisit the d-dimensional result and look for a similar rescaling. From Eq. (62) it is straightforward to verify that in this case

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\widetilde{a}_d \left(\frac{r_0(\epsilon)}{L}\right)^d \left(\frac{E_{\rm gap} - \epsilon}{\Delta_{\rm g}}\right)^{3/2}\right] \quad ,$$

where $\widetilde{a}_d$ represents some numerical constant, and r_0 is the characteristic length defined by Eq. (60). Finally, by defining

$$\widetilde{\Delta}_{\rm g}^{-3/2}(\epsilon) \equiv \frac{\delta}{\widetilde{\delta}(\epsilon)}\Delta_g^{-3/2},$$

where $\widetilde{\delta}(\epsilon) = 1/(\nu r_0^d(\epsilon))$ is the level spacing inside a region of size r_0, the volume dependent prefactor can be absorbed into the expression and DoS can be brought to the form

$$\frac{\nu(\epsilon < E_{\rm gap})}{\nu} \sim \exp\left[-\widetilde{a}_d \left(\frac{E_{\rm gap} - \epsilon}{\widetilde{\Delta}_{\rm g}}\right)^{3/2}\right] \quad ,$$

revealing a simple relation between the $d = 0$ and $d > 0$ problems.

3.5.2. *Universalities*

The coincidence of Eqs. (64) and (67) indeed suggests that gap fluctuations are universal. To understand why, let us consider the following: At first glance the actions (65) and (33), used in section 2.3.1 to describe the quantum dot problem, would seem not to have much in common. However, a simple and general argument may be established to reveal the universal character. As before, defining $\theta_{\rm mf}(\epsilon) = i\pi/2 + \phi_{\rm mf}(\epsilon)$, the mean-field DoS for the SN device is given by $\nu_{\rm mf}(\epsilon) = 2\nu\ {\rm Im}\ \sinh\phi_{\rm mf}(\epsilon)$, where $\phi_{\rm mf}$ is determined by the condition $\delta S/\delta\phi[\phi = \phi_{\rm mf}] = 0$. Since the DoS displays a square root singularity described by Eq. (63), the (saddle-point) action near the edge is constrained to be of the form

$$S[\hat{\phi}] = -k\ {\rm str}\left[\frac{1}{3}\hat{s}^3 + \left(\frac{\delta}{2\pi}\right)^2\left(\frac{\epsilon_+ - E_{\rm gap}}{\Delta_{\rm g}^3}\right)\hat{s}\right] \quad ,$$

where $\hat{s}(\epsilon) = \sinh\hat{\phi}(\epsilon) - \sinh\hat{\phi}_{\rm mf}(E_{\rm gap})$. Here, the elements $\hat{\phi} = {\rm diag}(\phi_{\rm bb}, \phi_{\rm ff})$ and $\hat{s} = {\rm diag}(s_{\rm bb}, s_{\rm ff})$ are diagonal in the superspace. (As one may check, a variation of the action for $\epsilon > E_{\rm gap}$ obtains the symmetric mean-field solution

$$s_{\rm bb} = s_{\rm ff} = i\frac{\delta}{2\pi}\sqrt{\frac{\epsilon - E_{\rm gap}}{\Delta_g^3}}$$

which in turn recovers the expression (63) for $\nu(\epsilon)$.) Moreover, since the term containing $\hat{s}$ is linear in the energy, we can determine the value of k from the knowledge that ϵ appears in the action as $(2\pi\epsilon_+/\delta)\sinh\hat{\phi}$. (It is this term that can more generally contain the Dyson index 'β', which therefore appears in the general expression for gap fluctuations described in Ref. [46].) In the present case, we thus have $k = (2\pi\Delta_{\rm g}/\delta)^3$.

Now, as discussed in the previous section, when $\epsilon < E_{\rm gap}$ there exists two saddle-point solutions at

$$s_\pm = \pm\frac{\delta}{2\pi}\sqrt{\frac{E_{\rm gap} - \epsilon}{\Delta_g^3}}.$$

As before, one of these solutions ($s_-(\epsilon) \rightsquigarrow \phi_{\rm mf}(\epsilon)$) is associated with the conventional symmetric mean-field solution while the other represents a second saddle-point accessible only to the bosonic contour. Taking this second, symmetry broken saddle-point into account (i.e. setting $s_{\rm bb} = s_+$ and $s_{\rm ff} = s_-$), one obtains the saddle-point action

$$S[\hat{\phi}] = \frac{4}{3}\left(\frac{E_{\rm gap} - \epsilon}{\Delta_{\rm g}}\right)^{3/2}.$$

It is this symmetry broken saddle-point which controls the sub-gap DoS and leads to the universal scaling form proposed in Ref. [46]. This generalizes the arguments applied to the superconducting dot with magnetic impurities.

3.5.3. *Discussion*

Following on from this discussion, to conclude this section, let us make two remarks which bear on the universality of the general scheme. The first of these remarks concerns the integrity of the scaling of the sub-gap DoS when different impurity distributions are taken into account. The second remark concerns the extension of the ideas above to the consideration of the hybrid SN system beyond the zero-dimensional regime.

Firstly, for the superconductor with magnetic impurities, one can generalize the arguments above to show that the energy scaling of the sub-gap DoS even in the d-dimensional case is insensitive to the nature of the random impurity distribution. This is in contrast to Lifshitz band tail states in semi-conductors where the energy scaling depends sensitively on this distribution. To understand this, let us suppose that the distribution of magnetic impurities $J\mathbf{S}(\mathbf{r})$ is not Gaussian δ-correlated, as we assumed throughout, but obeys some arbitrary statistics defined by a probability functional $P[J\mathbf{S}(\mathbf{r})]$. When the ensemble average over $J\mathbf{S}(\mathbf{r})$ is performed one would obtain in the Ψ-field action a contribution of the form

$$\ln \left\langle \exp\left[-i\int d\mathbf{r}\ \bar{\Psi}J\mathbf{S}\cdot\sigma^{\rm sp}\Psi\right]\right\rangle_P \equiv C[\bar{\Psi}\sigma^{\rm sp}\Psi(\mathbf{r})]\ ,$$

which defines $C[\cdots]$, the generating functional of connected correlators of $J\mathbf{S}(\mathbf{r})$. Though this is in general a very complicated and indeed non-local functional of $\bar{\Psi}\sigma\Psi(\mathbf{r})$, one can in principle find a *local* Q-field action by including pairings only at coincident points, justified by the assumption $(\ell/\xi)^d \ll 1$ about the non-magnetic disorder. The mean-field description of this system then follows from the homogeneous solution of the saddle-point equation, an Usadel equation like (50) with some potential. Generally this potential will have the same characteristics as the real potential of (55) plotted in Fig. 9 on the line $\mathrm{Im}\ \theta_1 = \pi/2$. The central maximum is due to the $|\Delta|$ term; the upturn at large ϕ arises from the small pair-breaking part, and the asymmetry comes from the ϵ term. Now, if mean-field theory leads to a square-root singularity in the DoS (a circumstance which can be avoided only by a special tuning of parameters), one can expect that increasing the energy leads to the maximum merging with one of the minima according to

$$V_{\rm r}[\phi] \simeq -\alpha\left(\frac{E_{\rm gap}-\epsilon}{|\Delta|}\right)^{1/2}(\phi-\phi_{\rm mf})^2 + \beta(\phi-\phi_{\rm mf})^3$$

with α and β chosen appropriately. Then the analysis of section 3.4 applies. In particular the scaling of the exponent with $((E_{\rm gap}-\epsilon)/|\Delta|)^{(6-d)/4}$ is expected to be *universal and independent of the details of the magnetic impurity potential*.

Now let us turn to the generality of the present scheme in describing ‘gap fluctuations’ in extended hybrid superconductor/normal systems. The latter has been discussed in a very recent paper by Ostrovsky *et al.* [61]. In this work, the authors developed an instanton approach analogous to that employed in the magnetic impurity system here to estimate the profile of gap fluctuations in the d-dimensional SNS system. Now from the discussion above, it is possible to expose the relation between these two works: in the SNS system, the energy gap induced in the normal region due to the proximity effect is determined by the Thouless energy defined as $E_{\rm T} \sim 1/\tau_{\rm dwell}$, where $\tau_{\rm dwell}$ is the time required for electrons in the normal region to feel the presence of the superconductor [33]. The Thouless energy is determined by $E_{\rm T} \sim \min\{D/L^2, \Gamma N\delta\}$ where Γ is the transparency of the contact to the superconductor ($\Gamma = 1$ in the analysis of the zero-dimensional system above).

In the diffusive limit $D/L^2 \ll \Gamma N\delta$, at the mean-field level, the position of the quasi-particle energy gap is found by solving the Usadel equation with the appropriate boundary conditions [62]. As a result one obtains a square root singularity in the DoS. In this case the mean-field solution is itself inhomogeneous. The sub-gap correction is found by identifying a second inhomogeneous instanton configuration that breaks supersymmetry at the level of the action [61]. Both solutions merge at the mean-field gap. Now, following the arguments above, it is simple to see how the phenomenology of Ref. [61] fits into the same general scheme: in this case the relevant coordinate of Q interpolates between the inhomogeneous mean-field solution and the instanton. The result is a sub-gap DoS which assumes the familiar form of Eq. (67), with appropriately defined geometry dependent parameters $E_{\rm gap}$ and $\Delta_{\rm g}$. Naturally the introduction of $d_\perp$ transverse dimensions gives the expected energy dependence of $(E_{\rm gap}-\epsilon)^{(6-d_\perp)/4}$ in the exponent.

In the opposite limit $D/L^2 \gg \Gamma N\delta$ (not considered in Ref. [61]), gradients of Q are heavily penalized and the coupling to the lead *must* be retained in its 'logarithmic form', with Q being taken as constant in the dot. (Indeed, the logarithm is crucial to reproduce even the mean-field expression for the DoS (63) with the correct coefficients.) This is the true zero-dimensional limit treated above. As we have seen, with this action, one recovers the known universal expression for the spectrum of gap fluctuations below the mean-field edge. Finally, it is interesting to note that the parallel (at the mean field level) between the magnetic impurity problem and proximity effect situations was noted a long time ago [37, 63].

So far we have uncovered a high degree of universality in our discussion of sub-gap states. This parallels the well-appreciated fact that the AG theory recurs in a great many pair-breaking scenarios (see the 'equivalence theorems' of Ref. [37] and Ref. [64] for a more unusual context). In the next section we will discuss two of the classical realizations of AG and ask whether the equivalence extends to description of sub-gap states

3.6. SUB-GAP STATES IN OTHER REALIZATIONS OF THE AG THEORY

Evidently, time-reversal symmetry can be broken by an external perturbation in a number of ways. Soon after the seminal work of AG it was understood [65, 66] that the single quasi-particle Green's function of a thin dirty film in a parallel field is described by the AG theory. Thus the predictions for both T_c and the DoS are the same as before. In fact, the DoS of the parallel field system is experimentally better described by the AG theory than the magnetic impurity system. This has been blamed on weaknesses of the model (37) in the magnetic impurity case [4]. We now analyze the parallel field system to address the universality of the sub-gap structure.

3.6.1. *Thin Film Superconductor in an in-plane Magnetic Field*
Consider a thin film of thickness $d_\perp$ in the $x-y$ plane and in the limit where $d_\perp \ll \xi, \lambda_L$. Here λ_L is the London penetration depth. The effective action for this system reads

$$S = -\frac{\pi\nu}{8}\int \mathrm{str}\left[D(\widetilde{\nabla}Q)^2 - 4i(\epsilon\sigma_3^{\mathrm{cc}} + |\Delta|\sigma_2^{\mathrm{ph}})\sigma_3^{\mathrm{ph}}Q\right] .$$

Recall $\widetilde{\nabla} \equiv \nabla - ie\mathbf{A}[\sigma_3^{\mathrm{ph}}, \,]$ is the covariant derivative. To analyze the saddle-point structure, note that in London gauge we have $A_y = Hz$, and this also implies that Q_{sp} is constant across the film. Since

$$\frac{1}{d_\perp}\int dz\, A(z) = 0, \qquad \frac{1}{d_\perp}\int dz\, A^2(z) = \frac{1}{12}(Hd_\perp)^2 ,$$

we see that the paramagnetic piece must vanish from the saddle-point equation and we are left with

$$D\nabla(Q\nabla Q) + i\left[Q, \epsilon\sigma_3^{\mathrm{ph}} \otimes \sigma_3^{\mathrm{cc}} + i\Delta\sigma_1^{\mathrm{ph}}\right] + \frac{1}{2\tau_H}\left[Q, \sigma_3^{\mathrm{ph}}Q\sigma_3^{\mathrm{ph}}\right] = 0 ,$$

where $1/\tau_H = D(Hd_\perp)^2/6$, and we understand that the spatial derivatives are in the plane. We note that this coincides with the Usadel equation (48) for the magnetic impurity problem, once the spin singlet ansatz is made there. Evidently the results of the AG theory are carried over wholesale with τ_H for τ_s. Furthermore, it is not difficult to see that the description of the sub-gap states is also identical, with the instantons living in the $x-y$ plane, where they are not disturbed by the z-dependence of $\mathbf{A}$.

In the introduction to this chapter we referred to intrinsic and extrinsic pictures of tail state formation. It is clear that there is no extrinsic picture for the present situation like the ‘droplet’ view of the magnetic impurity system. Tail state formation is more likely due to a proportion of the spectrum being anomalously lacking in phase rigidity.

Before moving on to an example where this universality is not repeated, it should be noticed that the parallel field and magnetic impurity problems are not equivalent as far as spin related physical quantities (like the spin susceptibility) are concerned [37], for obvious reasons. In the same way, if we drive a thin film into the gapless phase by applying a parallel field, it is clear on symmetry grounds alone that the spectral and transport properties of the low-energy quasi-particles are those of class C. This is in fact a thermal (and spin) *insulator* — see [35].

3.6.2. *Pair breaking by supercurrent*

Up to now we have not needed to discuss the phase of the order parameter. To describe the situation where a supercurrent flows we will need to take this into account. Varying the action with respect to Q, an applying the Ansatz,

$$Q_{\rm sp} = \left[\sigma_3^{\rm cc} \otimes \sigma_3^{\rm ph} \cosh \hat{\theta} + i \sinh \hat{\theta} (\sigma_1^{\rm ph} \cos \phi - \sigma_2^{\rm ph} \sin \phi)\right] \otimes \mathbb{1}^{\rm sp} ,$$

where ϕ will coincide with the phase of the order parameter $\Delta = |\Delta| e^{i\phi}$ at a saddle-point, the saddle-point equations obeyed by $\hat{\theta}$ and ϕ divide into two. With $\mathbf{A} = 0$,

$$\begin{aligned} \nabla^2_{\mathbf{r}/\xi} \hat{\theta} + 2i \left(\cosh \hat{\theta} - \frac{\epsilon}{|\Delta|} \sinh \hat{\theta}\right) - \frac{D}{2} (\nabla_{\mathbf{r}/\xi} \phi)^2 \sinh(2\hat{\theta}) = 0 \ , \\ \nabla_{\mathbf{r}/\xi} (\sinh \hat{\theta} \nabla_{\mathbf{r}/\xi} \phi) = 0 \ . \end{aligned} \tag{68}$$

The second equation describes the conservation of (spectral) current. Evidently a constant phase gradient (that arises when we apply a phase difference across a piece of uniform superconducting wire, say) acts as a pair-breaking perturbation and the AG mean-field result holds as before with $\zeta = D(\nabla \phi)^2 / 2|\Delta|$.

Do localized tail states form in this situation? On the basis of the first equation (68) we would be inclined to think so. But the current conservation law adds a complication. This can be dealt with in 1D by substituting

$$\partial_x \phi = \frac{\text{const}}{\sinh^2 \hat{\theta}} \ . \tag{69}$$

It is straightforward to see that the resulting potential does not allow for the existence of bounce solutions that we discussed before: there are no localized sub-gap states! On reflection this makes sense: a localized state can not carry a supercurrent, so there would be no pair-breaking effect.

Thus we see that the description of sub-gap states may not have quite the universality of the AG mean-field solution for the extended part of the spectrum. To finish we mention one slightly unusual example where the tail state formation *does* proceed in the same way. This is the case of gap fluctuations in superconductors with a quenched inhomogeneous distribution of the BCS coupling constant [67], where a mean-field description identical to AG was given in [64]

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