Localization from σ-model geodesics

A. Lamacraft
Department of Physics, Princeton University, Princeton, NJ 08544, USA[*]

B. D. Simons
Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, UK

M. R. Zirnbauer
Institut für Theoretische Physik, Universität zu Köln, Köln, Germany

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We use a novel method based on the semi-classical analysis of σ-models to describe the phenomenon of strong localization in quasi one-dimensional conductors, obtaining the density of transmission eigenvalues. For several symmetry classes, describing random superconducting and chiral Hamiltonians, the target space of the appropriate σ-model is a (super)group manifold. In these cases our approach turns out to be exact. The results offer a novel perspective on localization.

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I. INTRODUCTION

Beginning with the early work of Thouless[1,2], quasi one-dimensional conductors have provided a valuable arena in which to explore the influence of quantum interference effects on the transport properties of weakly disordered phase coherent conductors. By the early 80's, a complete scaling theory of localization in multi-mode wires had been formulated by Dorokhov[3] (and developed later by Mello, Pereyra and Kumar[4]) as a Brownian motion of eigenvalues of the transmission matrix — the 'DMPK equation' (for a recent review see, e.g., Ref. 5). For several symmetry classes, analytic results for the low moments of conductance were obtained in both the metallic and strongly localized regimes[5,6,7]. Lately, a third, and potentially more versatile approach has been developed to investigate quantum transport in (multi-terminal) disordered conductors. By formally relating the transmission eigenvalue distribution to a 'multi-component Green's function' (see below), Nazarov has shown[8] that known results in the metallic limit can be inferred from the equations of motion of the average quasi-classical Green function[9,10,11].

The transport properties of a conductor are fully specified by the transmission matrix $\mathbf{t}$[12]. In particular, the Landauer-Büttiker formula allows the conductance to be expressed through eigenvalues $\mathcal{T}_n$ of the 'squared' transmission matrix,

$$G = \frac{e^2}{h}\mathrm{tr}\,\mathbf{t}^\dagger\mathbf{t} = \frac{e^2}{h}\sum_n \mathcal{T}_n\,.$$

The statistics of the $\mathcal{T}_n$ follow the universal Dorokhov distribution, valid in the metallic limit (i.e. where the dimensionless conductance $\mathfrak{g} \equiv hG/e^2 \gg 1$):

$$\rho(\mathcal{T} \gtrsim e^{-2L/\ell}) = \Big\langle \sum_n \delta(\mathcal{T}-\mathcal{T}_n)\Big\rangle = \frac{\mathfrak{g}}{2}\frac{1}{\mathcal{T}\sqrt{1-\mathcal{T}}}\,. \quad (1)$$

By solving the corresponding quasi-classical equation in the diffusive limit, Eq. (1) has been established under very general conditions. Yet, while the method introduced by Nazarov is capable of describing accurately the transmission eigenvalue distribution in the metallic phase, the method fails to account for the phenomena of weak and strong localization.

In recent years, the development of a field theoretic technique to explore phase coherence phenomena in disordered conductors has exposed the strengths and limitations of the quasi-classical scheme. Weakly disordered conductors and superconductors can be classified into one of ten symmetry classes[13], their spectral and transport properties specified by a field theory of nonlinear σ-model type[14,15]. It was realized that within the formally exact framework provided by the field theory, the quasi-classical equations of motion represent a saddle-point or mean-field approximation, involving only a single field configuration[16,17]. Indeed, drawing on the substantial literature on the quasi-classical equations in the superconducting context (the Eilenberger[9] and Usadel[11] equations), this identification simplified substantially the analysis of physical realizations of the 'novel symmetry' classes of disordered superconductors introduced by Altland and Zirnbauer[15]. The absence of localization in the quasi-classical theory follows then from the neglect of fluctuations about this field configuration.

Describing localization within the σ-model is however far from trivial. While weak localization corrections to transport can be developed as a systematic perturbation theory, the transition to strong localization is signalled by the growth of contributions non-perturbative in the conductance. Indeed, a calculation by Rejaei[18] of the exact transmission eigenvalue distribution relied on a mapping of the σ-model for the quasi one-dimensional system onto a heat kernel (see also Ref. 19). Such an analysis affords little intuition into the origin of localization within the framework of the σ-model. In particular, can the transition (or, in quasi one-dimension, the crossover) to strong localization be understood in terms of certain σ-model field configurations?

A preliminary answer to this question was provided in an insightful work by Muzykantskii and Khmel'nitskii[16]. Using the field theoretic approach, it was demonstrated that the long-time response of a weakly disordered quasi one-dimensional wire to a voltage step was controlled by spatially inhomogeneous saddle-point field configurations of the σ-model action. Non-perturbative in the conductance, these field configurations were associated with rare "anomalously localized" states embedded deep within the metallic phase. Whether these states provided a caricature of states near the Anderson transition remains to date the subject of debate. Later, in a related activity, it was shown the nucleation of localized quasi-particle states inside the gap of a weakly disordered symmetry broken superconductor were ascribed to spatially inhomogeneous saddle-point configurations of the action[20].

In this paper we will show that, for a range of symmetry classes, the physics of strong localization is captured *exactly* by the inclusion of a set of new saddle points combined with their associated Gaussian fluctuations. This recalls a previous investigation by Andreev and Altshuler[21] of energy level correlations in symmetry broken (zero-dimensional) chaotic systems — random matrix ensembles belonging to the unitary symmetry class. There it was shown that the two-point correlator of the density of states could be fully recovered from the inclusion of two saddle points together with their associated Gaussian fluctuations. In both cases, the coincidence can be traced to the property of semiclassical exactness shared by the different σ-model field theories. The aim of the present paper is to elucidate this principle in the quasi one-dimensional system comparing results for transmission eigenvalue distributions to those obtained from heat kernel methods. The novel view of localization presented here may inform the treatment of more complicated problems.

Following Rejaei, our analysis rests on a relation which allows the ensemble average of the generating function of the transmission matrix $\mathbf{tt}^\dagger$ (at a particular energy) of a quasi one-dimensional sample

$$\mathcal{Z}(\phi,\theta) = \left\langle \frac{\det(1-\sin^2(\theta/2)\mathbf{tt}^\dagger)}{\det(1-\sinh^2(\phi/2)\mathbf{tt}^\dagger)} \right\rangle , \qquad (2)$$

to be presented as a partition function of a nonlinear σ-model of the appropriate symmetry class. The density of transmission eigenvalues may then be extracted using the relation

$$F(\phi) \equiv \frac{\partial}{\partial\theta}\mathcal{Z}(\phi,\theta)\big|_{\theta=i\phi} = \sum_n \left\langle \frac{-i\sinh\phi}{\cosh\phi_n + \cosh\phi} \right\rangle , \qquad (3)$$

where $\mathcal{T}_n = 1/\cosh^2(\phi_n/2)$ denote the eigenvalues of the matrix $\mathbf{tt}^\dagger$. From this function one can infer the transmission eigenvalue density through the relation

$$\rho(\phi) \equiv \sum_n \left\langle \delta(\phi-\phi_n) \right\rangle = \frac{1}{2\pi}\big(F(\phi+i\pi) - F(\phi-i\pi)\big) . \qquad (4)$$

The moments of the transmission matrix can be cast in terms of the retarded and advanced Green's functions $G^{R/A} \equiv (\epsilon \pm i\delta - H)^{-1}$ of the microscopic Hamiltonian of the wire as

$$\mathrm{tr}(\mathbf{tt}^\dagger)^n = \mathrm{tr}_{\mathbf{r}}(\hat{v}_L G^A_\epsilon \hat{v}_R G^R_\epsilon)^n ,$$

where $\hat{v}_{L/R}$ denotes the current operator through left and right cross-sections at $x=0$ and $x=L$ respectively, and the trace on the right hand side runs over spatial coordinates. Following Nazarov, it is helpful to note that the determinants in Eq. (2) can be recast in terms of a multi-component Green's function according to the relation

$$\begin{aligned}
\det(1-\gamma_1\gamma_2\mathbf{tt}^\dagger) &= \det(1-\gamma_1\gamma_2\hat{v}_L G^A_\epsilon \hat{v}_R G^R_\epsilon) \\
&= \det\begin{pmatrix} 1 & \gamma_1 G^R_\epsilon \hat{v}_L \\ \gamma_2 G^A_\epsilon \hat{v}_R & 1 \end{pmatrix} \\
&\propto \det\begin{pmatrix} \epsilon - H + i\delta & \gamma_1 \hat{v}_L \\ \gamma_2 \hat{v}_R & \epsilon - H - i\delta \end{pmatrix} ,
\end{aligned}$$

where, for example, in the case of the numerator of (2), $\gamma_1\gamma_2 \equiv \sin^2(\theta/2)$. Thus the generating function $\mathcal{Z}(\phi,\theta)$ may be related to the Green's function of an enlarged Hamiltonian, which includes an off-diagonal 'vector potential'.

With this representation in hand, the outline of the remainder of the paper is as follows. In the next section we present a complete analysis of the localization properties of Class CI, one of the superconducting symmetry classes introduced by Altland and Zirnbauer[15]. We outline the σ-model description of this symmetry class and reproduce Nazarov's calculation in this context, before introducing the full set of saddle-point trajectories - geodesics on the target space - and performing a complete semiclassical calculation. From the latter we will extract the transmission eigenvalue density and mean conductance. The exactness of the results obtained are then verified by heat kernel methods. In the following section, the techniques developed above will be applied to two additional symmetry classes that are amenable to such a treatment. Finally, in the last section, we will draw conclusions.

II. CLASS CI

Superconductors which exhibit both time reversal symmetry and spin rotation invariance belong to symmetry class CI. By introducing the familiar Nambu doublet of electron and hole operators in what we will refer to as the 'particle-hole' (ph) space, in the mean-field approximation, the quasi-particle Gor'kov or Bogoliubov-de Gennes Hamiltonian $\hat{\mathcal{H}}_{\rm mf}$ can be written as

$$\hat{\mathcal{H}}_{\rm mf} = \frac{1}{2}\sum_{ij} \begin{pmatrix} \hat{c}^\dagger_{i\uparrow} & \hat{c}_{i\downarrow} \end{pmatrix} \overbrace{\begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij} & -h_{ij} \end{pmatrix}}^{H_{ij}} \begin{pmatrix} \hat{c}_{j\uparrow} \\ \hat{c}^\dagger_{j\downarrow} \end{pmatrix} ,$$

where h_{ij} and Δ_{ij} represent real symmetric matrix elements corresponding to the non-interacting single-particle Hamiltonian of the quasi one-dimensional disordered wire and the superconducting order parameter respectively. With this structure, the matrix Hamiltonian H satisfies the symmetry relations

$$H = H^T = -\mathcal{C}H^T\mathcal{C}^{-1} \qquad \mathcal{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\rm ph} , \tag{5}$$

which in turn implies the identity

$$G^R_\epsilon = -\mathcal{C}(G^A_{-\epsilon})^T\mathcal{C}^{-1} ,$$

between the retarded and advanced Green's functions.

A. Partition function

With this definition, the ratio of determinants which specifies the generating function (2) can be conveniently expressed via a supersymmetric field integral, with the numerator arising from a fermionic integral and the denominator from its bosonic counterpart. Since these are not identical, the corresponding partition function will inherit boundary conditions which break the supersymmetry. To identify the soft diffusion modes of the superconducting system, it is convenient to effect a doubling of the field space of the supervectors ψ to include a 'charge-conjugation' (cc) space[22]

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi \\ \mathcal{C}\bar\psi^T \end{pmatrix}_{\rm cc} \qquad \bar\Psi = \frac{1}{\sqrt{2}}\left(\bar\psi, -\psi^T\mathcal{C}^{-1}\right)_{\rm cc}. \tag{6}$$

Focusing on the transmission eigenvalue distribution at $\epsilon = 0$, we notice that if $G^R_0 \sim \langle\psi\bar\psi\rangle$ is the $(1,1)$ component of the matrix Green's function $\mathcal{G}$ in the cc space, then the advanced component $G^A_0 \sim -\mathcal{C}\langle\bar\psi^T\psi^T\rangle\mathcal{C}^{-1}$ is just the (2,2) component. Thus the enlarged structure required is just the usual cc space. Altogether, this leads to the representation

$$\frac{\det(1-\gamma_1\gamma_2\mathbf{t}\mathbf{t}^\dagger)}{\det(1-\zeta_1\zeta_2\mathbf{t}\mathbf{t}^\dagger)} = \int \mathcal{D}\Psi\mathcal{D}\bar\Psi \exp\left[\frac{i}{2}\int d\mathbf{r}\bar\Psi \overbrace{\left[i\delta\Sigma_3 - \mathcal{H}\right]}^{\mathcal{G}^{-1}}\Psi\right]$$

where

$$\mathcal{H} = H\otimes\mathbb{1}_{\rm bf}\otimes\mathbb{1}_{\rm cc} + \hat v_L\Gamma_1\otimes\Sigma_+\otimes\mathbb{1}_{\rm ph} + \hat v_R\Gamma_2\otimes\Sigma_-\otimes\mathbb{1}_{\rm ph} , \tag{7}$$

where $\Gamma_{1,2} = {\rm diag}(\zeta_{1,2},\gamma_{1,2})_{\rm bf}$, and Σ_i denote the Pauli matrices in the cc space.

Once cast in the form of a functional field integral, it is a straightforward (if somewhat lengthy) and standard procedure to show that the low-energy properties of the partition function are contained with an effective field theory of nonlinear σ-model type (for a detailed discussion of the explicit derivation, we refer to one of the many standard references, e.g. Ref.[17,22]). Two points should be remarked upon. The first is that, since Class CI has two symmetries – particle-hole and time reversal (tr) symmetry (see Eq. 5) – the supervector needs to be doubled *twice*. The size of the supermatrix field $Q(\mathbf{r})$ that appears at an intermediate stage in the derivation is thus 16×16 (ph×cc×tr×bf), but this is quickly reduced to the 8×8 field q by $Q = \sigma_3 q$ on the saddle-point manifold (σ_i denote the Pauli matrices in ph space), with the ph space then disappearing from view. Secondly, following Rejaei, the 'vector potential' associated with the coupling of the wire to the external leads, can be absorbed into a rigid boundary condition on the field integral. Taking the metallic contacts at $x = 0, L$ to be to clean normal conducting leads, the ensemble averaged partition function assumes the form (with $\hbar = 1$)

$$\left\langle\frac{\det(1-\gamma_1\gamma_2\mathbf{t}_\epsilon\mathbf{t}^\dagger_\epsilon)}{\det(1-\zeta_1\zeta_2\mathbf{t}_\epsilon\mathbf{t}^\dagger_\epsilon)}\right\rangle = \int_{q(0)=\Sigma_3}^{q(L)=S\Sigma_3S^{-1}} \mathcal{D}q\exp\left(\frac{\pi\nu D}{8}\int d\mathbf{r}\,{\rm STr}\left[\nabla q\right]^2\right) , \tag{8}$$

where the field integral is over 8×8 supermatrix fields q subject to the nonlinear constraint $q^2(\mathbf{r}) = \mathbb{1}$. Finally, the supersymmetry breaking source terms enter the boundary condition, through the rotation matrix

$$S = \exp\left(i\Gamma_2\otimes\Sigma_-\otimes\mathbb{1}_{\rm tr}\right)\exp\left(i\Gamma_1\otimes\Sigma_+\otimes\mathbb{1}_{\rm tr}\right) .$$

Specifically, if we choose

$$\gamma_1 = \frac{1}{2}\sin\theta , \qquad \gamma_2 = \tan\theta/2$$
$$\zeta_1 = \frac{i}{2}\sinh\phi , \qquad \zeta_2 = i\tanh\phi/2 ,$$

then the boundary condition at $x = L$ is $q(L) =$

$\mathrm{diag}(q_{\mathrm{bb}}(L), q_{\mathrm{ff}}(L))_{\mathrm{bf}}$ where

$$q_{\mathrm{bb}}(L) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ -\sinh\phi & -\cosh\phi \end{pmatrix}_{\mathrm{cc}} \otimes \mathbb{1}_{\mathrm{tr}} \; ,$$
$$q_{\mathrm{ff}}(L) = \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}_{\mathrm{cc}} \otimes \mathbb{1}_{\mathrm{tr}} \; , \qquad (9)$$

and we have obtained the required form of Rejaei's relation (2). An alternative approach that matches Nazarov's orginal formulation is to use the explicit expression for $F(\phi)$ that follows from Eq. (3) and relate it directly to the Green's function in the cc space, i.e.

$$\begin{aligned} F(\phi) &= -\frac{i}{2}\sinh\phi \left\langle \mathrm{tr}\left[\mathbf{t}\mathbf{t}^\dagger \left(1+\sinh^2(\phi/2)\mathbf{t}\mathbf{t}^\dagger\right)^{-1}\right]\right\rangle \\ &= \left\langle \mathrm{tr}_{\mathbf{r}}\left[\hat{v}_R \mathcal{G}^{12}_{\mathrm{bb}}\right]\right\rangle \big|_{\theta=i\phi} \\ &= -\frac{i\pi\nu D}{2}\mathcal{A}\left\langle \mathrm{tr}_{\mathrm{tr}}\left(q\partial_x q\right)^{12}_{bb}\big|_{x=0}\right\rangle_q \big|_{\theta=i\phi}, \qquad (10) \end{aligned}$$

where $\langle\cdots\rangle_q$ denotes an average with respect to the σ-model action (8), $\mathcal{A}$ is the cross-sectional area, and ν is the single particle density of states. $\mathrm{tr}_{\mathrm{tr}}$ denotes the trace over the tr space. Henceforth, we will work only with the one-dimensional field $q(x)$, assuming that the width of the wire is much smaller than the localization length.

B. Nazarov's calculation

To develop a theory of localization from the field integral, it is first useful to establish contact with the theoretical framework introduced by Nazarov. As discussed in the introduction, the quasi-classical theory formulated by Nazarov is contained within the saddle-point structure of the present action: Specifically, a variation of the action with respect to q subject to the nonlinear constraint, $q^2 = \mathbb{1}$, obtains the saddle-point equation

$$\partial_x(q\partial_x q) = 0,$$

which expresses conservation of the matrix current $q\partial_x q$. Indeed, associating q with the average quasi-classical Green function, the saddle-point equation can be interpreted as the generalization of the quasi-classical equation derived in Nazarov's early work to the superconducting wire.

The simplest solution satisfying the given boundary conditions is block diagonal in the bf space and takes the form

$$q_{\mathrm{bb}}(x) = \begin{pmatrix} \cosh\frac{\phi x}{L} & \sinh\frac{\phi x}{L} \\ -\sinh\frac{\phi x}{L} & -\cosh\frac{\phi x}{L} \end{pmatrix}_{\mathrm{cc}} \otimes \mathbb{1}_{\mathrm{tr}},$$
$$q_{\mathrm{ff}}(x) = \begin{pmatrix} \cos\frac{\theta x}{L} & -i\sin\frac{\theta x}{L} \\ i\sin\frac{\theta x}{L} & -\cos\frac{\theta x}{L} \end{pmatrix}_{\mathrm{cc}} \otimes \mathbb{1}_{\mathrm{tr}} \; , \qquad (11)$$

If one inserts this solution into Eq. (10), and neglects field fluctuations around the saddle point, one finds

$$F(\phi) = -i\pi\nu D\mathcal{A}\phi/L.$$

Then, making use of Eq. (4), one can straightforwardly recover the Dorokhov distribution (1) for a wire of dimensionless conductance $\mathfrak{g} = 2\pi\nu DA/L$.

However, when derived from the supersymmetric field theory, one can see that the saddle-point equation presents not just one solution, but a whole family of solutions in the compact fermionic sector

$$q^{(n)}_{\mathrm{ff}}(x) = \begin{pmatrix} \cos\frac{(\theta+2\pi n)x}{L} & -i\sin\frac{(\theta+2\pi n)x}{L} \\ i\sin\frac{(\theta+2\pi n)x}{L} & -\cos\frac{(\theta+2\pi n)x}{L} \end{pmatrix}_{\mathrm{cc}} \otimes \mathbb{1}_{\mathrm{tr}}, \quad n \in \mathbb{Z}. \qquad (12)$$

Thus we see that even at the level of a purely *semi-classical* analysis, Nazarov's treatment is missing two elements; corrections associated with fluctuations around the conventional saddle-point solution (11), and solutions (12) that loop multiply around the compact fermionic sector. We will show that these two 'channels' of corrections are, respectively, responsible for the phenomenon of weak localization — as, indeed, one might have guessed from the connection between the role of fluctuations in the σ-model and diagrammatic perturbation theory — and strong localization.

To uncover this relation, in the following we will undertake the semi-classical analysis taking into account fluctuations of the matrix fields at quadratic order around both the conventional and the non-trivial saddle-point field configurations.

C. Semi-classical calculation

To perform the field integration over the fluctuations, it is necessary to review the structure of the target manifold. For the minimal ($n = 1$) nonlinear σ-model associated with the symmetry class CI, the target space — spanned by matrices $q = w\Sigma_3 w^{-1}$ — turns out[13] to be in one-to-one correspondence with $G = \widetilde{\mathrm{OSp}}(2|2)$, by which we mean the Lie group $\mathbb{R}_+ \times \mathrm{SU}(2)$ extended to an orthosymplectic Lie supergroup. This group structure of the target manifold is the reason for the exactness of the semi-classical approximation[23,24].

The correspondence between q and $g \in G$ is described in the Appendix. The space of matrices (9) parameterized by ϕ and θ corresponds to the maximal Abelian subgroup $A \in G$

$$a = \mathrm{diag}(e^{\phi}, e^{-\phi}, e^{i\theta}, e^{-i\theta}) \in A \; ,$$

while, in terms of group elements $g \in G$, the generating function is

$$\mathcal{Z}(\phi,\theta) = \int_{g(0)=\mathbf{1}}^{g(T)=a(\phi,\theta)} \mathcal{D}g \, \exp\left(-\frac{1}{8}\int_0^T \mathrm{STr}\left[g^{-1}\dot{g}\right]^2 \, dt\right) . \qquad (13)$$

Here we have switched to the dimensionless variables $t \equiv x/\xi$, $T \equiv L/\xi$, where $\xi \equiv 2\pi\nu D\mathcal{A}$. (Note that, with this definition, $T = 1/\mathfrak{g}$.) With this parametrization, the set of saddle-point solutions discussed in the previous section corresponds to the geodesic trajectories of free particle motion

$$a_t^{(n)}(\phi,\theta) = \mathrm{diag}(e^{\phi t/T}, e^{-\phi t/T}, e^{i\theta_n t/T}, e^{-i\theta_n t/T}), \quad (14)$$

where $\theta_n = \theta + 2\pi n$ with $n \in \mathbb{Z}$. These configurations are associated with the classical action

$$S_{\mathrm{cl}}^{(n)} = \frac{1}{8}\int_0^T \mathrm{STr}[(a_t^{(n)})^{-1}\dot{a}_t^{(n)}]^2\, dt = (\phi^2 + \theta_n^2)/4T$$

To evaluate the contribution from Gaussian fluctuations, it is helpful to set $g_t = a_t^{(n)}\tilde{g}_t$, and express the element $\tilde{g}_t$ through the exponential parametrization $\tilde{g}_t \equiv \exp X_t$, with boundary conditions $X_0 = 0 = X_T$. Matrices X satisfying the osp(2|2) Lie algebra condition $X = -\varepsilon X^{\mathrm{T}}\varepsilon^{-1}$ with $\varepsilon = \mathrm{diag}(\sigma_x, i\sigma_y)_{\mathrm{bf}}$ may be written as

$$X = \begin{pmatrix} d & 0 & \alpha & \beta \\ 0 & -d & \gamma & \delta \\ \delta & \beta & e & b \\ -\gamma & -\alpha & c & -e \end{pmatrix}. \quad (15)$$

Then, to get the desired target space with the non-compact boson-boson sector $M_{\mathrm{bb}} = \mathbb{R}_+$, and compact fermion-fermion sector $M_{\mathrm{ff}} = \mathrm{SU}(2)$, one must take the variable d to be real, the variable e to be imaginary, and the complex variables b and c to be related by the condition $c = -\bar{b}$. The functional integration measure is trivial, $\mathcal{D}g = \mathcal{D}\tilde{g} = \mathcal{D}X + \ldots$, up to corrections (of order X^3) that will not affect the present calculation.

Using the parametrization for X given in Eq. (15) one obtains the quadratic action

$$\begin{aligned} S_{\mathrm{q}} &= \frac{1}{8}\int dt\, \mathrm{STr}\left(\dot{X}^2 + \dot{X}[(a_t^{(n)})^{-1}\dot{a}_t^{(n)}, X]\right) \\ &= \frac{1}{4}\int dt\, \Big(\dot{d}^2 - \dot{e}^2 - \dot{b}\dot{c} - \frac{i\theta_n}{T}\left(b\dot{c} - \dot{b}c\right) \\ &\qquad + 2\dot{\alpha}\dot{\delta} + \frac{\phi - i\theta_n}{T}(\alpha\dot{\delta} - \dot{\alpha}\delta) \\ &\qquad + 2\dot{\gamma}\dot{\beta} - \frac{\phi + i\theta_n}{T}(\gamma\dot{\beta} - \dot{\gamma}\beta)\Big)\ . \end{aligned}$$

Performing the Gaussian functional integral over d and e is the same as computing the kernel for free particle motion on the two-dimensional Euclidean space Lie A, and the result is $\mathrm{Det}^{-1}(-\partial_t^2)$. Performing the remaining Gaussian functional integrals gives determinants in the denominator (bosons: $b, c = -\bar{b}$) and numerator (fermions: α, δ and γ, β). Collecting all the determinants, one finally obtains the total fluctuation contribution to the partition function,

$$\frac{\mathrm{Det}\left(-\partial_t^2 - \frac{\phi + i\theta_n}{T}\partial_t\right)\mathrm{Det}\left(-\partial_t^2 + \frac{\phi - i\theta_n}{T}\partial_t\right)}{\mathrm{Det}\left(-\partial_t^2\right)\mathrm{Det}\left(-\partial_t^2 + \frac{2i\theta_n}{T}\partial_t\right)}\ .$$

The determinants are to be evaluated on the Hilbert space of square-integrable functions $L^2([0,T])$ with Dirichlet boundary conditions. If z is some complex number, the operator $D_z = -\partial_t^2 + 2(z/T)\partial_t$ on that Hilbert space has eigenfunctions $\sin(k\pi t/T)\, e^{zt/T}$ $(k \in \mathbb{N})$, with eigenvalues $\left((k\pi)^2 + z^2\right)/T^2$, so its (unregularized) determinant is given by

$$\mathrm{Det} D_z = \prod_{k=1}^{\infty}\left((k\pi)^2 + z^2\right)/T^2\ .$$

Taking the logarithmic differential (to kill the z–independent infinity) one obtains

$$\begin{aligned} \delta \ln \mathrm{Det} D_z &= \sum_{k=1}^{\infty}\frac{2z\delta z}{(k\pi)^2 + z^2} \\ &= \left(-\frac{1}{z} + \sum_{k\in\mathbb{Z}}\frac{z}{(k\pi)^2 + z^2}\right)\delta z \\ &= \left(-z^{-1} + \coth z\right)\delta z = \delta(\ln\sinh z - \ln z)\ . \end{aligned}$$

Therefore,

$$\mathrm{Det} D_z = \frac{\sinh z}{z} \times \mathrm{Det} D_0\ ,$$

and the above ratio of four determinants gives

$$\frac{\sinh\left(\frac{1}{2}(\phi + i\theta_n)\right)}{\frac{1}{2}(\phi + i\theta_n)}\,\frac{\sinh\left(\frac{1}{2}(\phi - i\theta_n)\right)}{\frac{1}{2}(\phi - i\theta_n)}\left(\frac{\sin\theta_n}{\theta_n}\right)^{-1}\ .$$

Finally, when combined with the exponential of the classical action and summed over n, one obtains the following expression for the partition function:

$$\begin{aligned} \mathcal{Z}_T(\phi,\theta) = \sum_{n\in\mathbb{Z}}\frac{\sinh\left(\frac{1}{2}(\phi + i\theta_n)\right)}{\frac{1}{2}(\phi + \mathrm{i}\theta_n)}\,\frac{\sinh\left(\frac{1}{2}(\phi - i\theta_n)\right)}{\frac{1}{2}(\phi - i\theta_n)} \\ \times \frac{\theta_n}{\sin\theta_n}\, e^{-(\phi^2 + \theta_n^2)/4T}\ . \quad (16) \end{aligned}$$

Note that the operator $-\partial_t^2 + (2i\theta_n/T)\partial_t$ starts to exhibit negative eigenvalues as soon as $|\theta_n|$ exceeds π. This means that the Gaussian functional integral over b and $c = -\bar{b}$ does not exist in those cases. Thus what we have done — evaluating the functional integral in the Gaussian approximation around all of the saddle points (without paying attention to question of existence), and summing contributions — was a purely formal calculation. Nevertheless, the answer obtained in this way turns out to be exact! Indeed, the validity of this expression can be established both directly and indirectly. In the following, we will motivate this conclusion by investigating the transmission eigenvalue density and the mean conductance from the partition function. Then, by interpreting the field integral as a heat kernel, we will find an alternative exact method of computation.

D. Eigenvalue density

With the partition function in hand, it is a straightforward matter to compute the eigenvalue density making use of the relations (2)-(4),

$$\rho_T^{\mathrm{CI}}(\phi) = \frac{1}{2T} - \frac{1}{2(\phi^2+\pi^2)} - \sum_{n\neq 0} \frac{e^{-n(n+1)\pi^2/T}}{2\pi^2 n} \,\mathrm{Re}\, \frac{\phi + i\pi(2n+1)}{\phi + i\pi(n+1)}\, e^{i\pi n\phi/T} \;. \qquad (17)$$

The first term of this expression is just the Dorokhov distribution, and originates from the $n = 0$ (Nazarov's) configuration. In this context, the second term simply denotes the weak localization correction associated with the $n = 0$ configuration. All higher-order terms of the perturbation expansion in powers of $T = 1/\mathfrak{g}$ turn out to vanish identically, a direct manifestation of semi-classical exactness. Indeed, the remaining terms in the result (17) depend non-analytically on T as $e^{-1/T}$, and arise from the non-trivial saddle-point configurations.

A non-trivial check on the validity of the expression follows from the requirement that $\rho_T(0) = 0$ for all $T > 0$. (One can infer this condition from the joint probability density given by Brouwer *et al.*[6].) The plot of the density function $\rho_T^{CI}(x)$ in Fig. 1 shows the characteristic 'crystallization' of transmission eigenvalues associated with the onset of localization. The largest $\mathcal{T}_n$ (smallest ϕ_n) eigenvalue crystallizes at $\phi = 2T$, corresponding to $\mathcal{T} \sim 4e^{-2T} = 4e^{-2L/\xi}$ if $L \gg \xi$.

E. Mean conductance

A second, and more direct way of inferring exponential localization is to investigate the mean (spin) conductance of the superconducting wire. The mean conductance $\langle \sum \mathcal{T}_n \rangle \equiv C(T)$ of the thick disordered wire can be obtained by integrating $\mathcal{T} = 1/\cosh^2(\phi/2)$ against the eigenvalue density $\rho_t(\phi)d\phi$ with the result

$$C(T) = \frac{1}{T} - \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{T} + \frac{2}{\pi^2 n^2}\right) e^{-n^2\pi^2/T} \;. \qquad (18)$$

On applying the rescaling $C(T) \to 4C(4T)$, the latter agrees with the expression derived in Ref. 6. The leading term T^{-1} is the Ohmic contribution, while the constant $-1/3$ represents the weak localization correction.

To extract the large-T asymptotics and demonstrate exponential localization, the above expression for $C(T)$ is inconvenient. Instead, it is convenient to implement an integral transform of the expression which brings it to a form in which the large-T asymptotics can be developed. Making use of the identities $(1+2/x)\, e^{-1/x} = -\int (1/x^2 - 2/x^3)\, e^{-1/x} dx$ and $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, one may easily show that

$$C(T) = \sum_{n\in\mathbb{Z}} \int_T^{\infty} (1/\tau^2 - 2\pi^2 n^2/\tau^3)\, e^{-n^2\pi^2/\tau} d\tau \;.$$

Noting that the function $k \mapsto (1/\tau^2 - k^2/2\tau^3)\, e^{-k^2/4\tau}$ is the Fourier transform of $x \mapsto 2x^2 e^{-x^2\tau}/\sqrt{\pi\tau}$, Poisson summation yields

$$C(T) = \frac{4}{\sqrt{\pi}} \sum_{l=1}^{\infty} l^2 \int_T^{\infty} e^{-l^2\tau} \frac{d\tau}{\sqrt{\tau}} \;.$$

Finally, introducing the complement of the error function,

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2} ds = 1 - \mathrm{erf}(z) \;,$$

the expression for $C(t)$ may be brought to the final form

$$\begin{aligned} C(T) &= \frac{1}{T} - \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{T} + \frac{2}{\pi^2 n^2}\right) e^{-n^2\pi^2/T} \\ &= 4 \sum_{l=1}^{\infty} l\, \mathrm{erfc}(l\sqrt{T}) \;. \end{aligned} \qquad (19)$$

While the original expression provides access to the small-T asymptotics, the second one gives easy access to the large-T behaviour, viz.

$$C(T) \overset{T\to\infty}{\longrightarrow} \frac{4}{\sqrt{\pi T}}\, e^{-T} \;.$$

Recalling $T = L/\xi$, the latter shows the characteristic exponential dependence of localization with a localization length ξ.

Consideration of the eigenvalue density and mean conductance provide strong circumstantial evidence that the expression derived for the generating function is exact. However, to emphasize the validity of the semi-classical expansion, it is useful to provide an explicit calculation of the partition function as a solution of the Euclidean-time 'Schrödinger' equation or heat kernel.

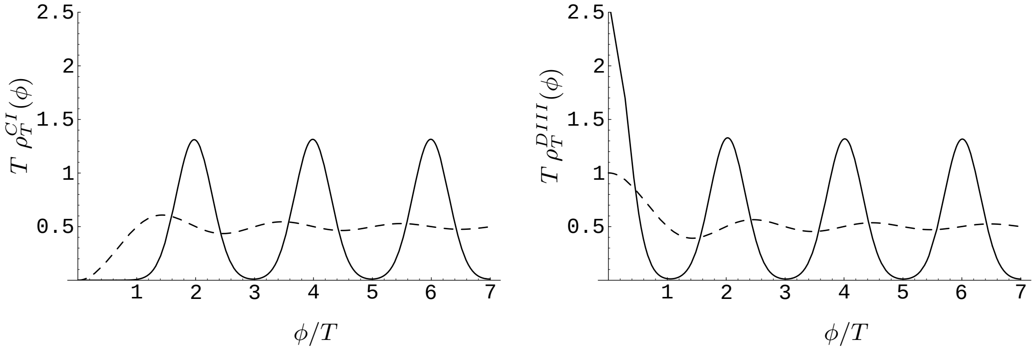

FIG. 1: Transmission eigenvalue densities for the classes CI and DIII at $T = 0.01$ (weak localization – dashed line) and $T = 20$ (strong localization – solid line). Note that although eigenvalue 'crystallization' occurs in both classes, localization in class DIII is absent due to the persistence of an eigenvalue near $\phi = 0$ (or $\mathcal{T} = 1$).

F. Heat Kernel

By canonical quantization, the path integral with the Lagrangian $\frac{1}{8}\mathrm{STr}\,(g^{-1}\dot{g})^2$ becomes the 'Schrödinger equation' for free-particle quantum motion on G. The Schrödinger operator is the negative of the Laplace-Beltrami operator $\triangle$ (or Laplacian for short) on G. Thus the heat kernel satisfies a Schrödinger-like equation which is the diffusion (or heat) equation on G:

$$\partial_t W(g,g';t) = \triangle W(g,g';t) \qquad (t>0)\ ,$$

where $\triangle$ acts on the left argument of W. The reduction from the full heat kernel to the radial function $\omega_t(\phi,\theta)$ on the maximal Abelian subgroup A takes the diffusion equation into

$$\partial_t \omega_t(\phi,\theta) = (\triangle\omega_t)(\phi,\theta) \qquad (t>0)\ ;$$

where we continue to denote the radial part of the Laplacian by $\triangle$ for simplicity. Since G has superdimension zero (there being the same number of bosonic and fermionic degrees of freedom) the short-time asymptotics of the heat kernel is given by

$$\omega_t(\phi,\theta) \stackrel{t\to 0+}{\longrightarrow} e^{-(\phi^2+\theta^2)/4t}\ .$$

(In dimensions $d \neq 0$, the Gaussian would be preceded by a factor $(4\pi t)^{-d/2}$.)

To proceed, we must draw on some basic facts from the geometry and analysis of the supermanifold G:

- Diagonalization of matrices g (i.e. $g = hah^{-1}$ with $a \in A$) defines a polar decomposition of G. By this decomposition, the G-invariant Berezin integration measure for G determines a radial integration measure $Jdxdy$, which is positive on the Weyl chamber chamber $[0,\infty]\times[0,\pi] \subset \mathrm{Lie}A$.

- By a standard formula[25] from the theory of Lie groups and symmetric spaces (for a recent pedagogical review see, for example, Ref. 26), the measure function J for the case at hand is given by

$$J = \frac{\sin^2\theta}{\sinh^2\left((\phi+i\theta)/2\right)\sinh^2\left((\phi-i\theta)/2\right)}\ .$$

- Finally, the Laplacian on radial functions is given by

$$(\triangle\omega_t)(\phi,\theta) = \left(J^{-1}\partial_\phi J\partial_\phi + J^{-1}\partial_\theta J\partial_\theta\right)\omega_t(\phi,\theta)\ .$$

By introducing complex coordinates $z = (\phi+i\theta)/2$ and $\bar{z} = (\phi-i\theta)/2$ it is apparent that the analytic square root of the measure function,

$$J^{1/2} = \frac{\sin(i\bar{z}-iz)}{\sinh(z)\sinh(\bar{z})} = i\frac{\cosh(z)}{\sinh(z)} - i\frac{\cosh(\bar{z})}{\sinh(\bar{z})}\ ,$$

is harmonic: $(\partial_\phi^2+\partial_\theta^2)J^{1/2} = 0$. Using this property one can easily verify that the radial part of the Laplacian can be cast in the form

$$(\triangle\omega_t)(\phi,\theta) = J^{-1/2}(\partial_\phi\partial_\phi+\partial_\theta\partial_\theta)J^{1/2}\omega_t(\phi,\theta)\ .$$

It therefore follows that the product $E_t = J^{1/2}\omega_t$ satisfies the Euclidean heat equation

$$\partial_t E_t = (\partial_\phi^2+\partial_\theta^2)E_t\ .$$

The Euclidean heat kernel in two dimensions is known to be $(4\pi t)^{-1}e^{-(\phi^2+\theta^2)/4t}$. However, this is not the solution we want here: As mentioned earlier, the heat kernel $\omega_t(\phi,\theta)$ is subject to zero-dimensional small-t asymptotics, $\omega_t(\phi,\theta)\to e^{-(\phi^2+\theta^2)/4t}$. This short-time behavior is achieved with the choice

$$\tilde{E}_t = \frac{4\theta}{\phi^2+\theta^2}\,e^{-(\phi^2+\theta^2)/4t}\ .$$

To confirm that $\tilde{E}_t$ satisfies the Euclidean heat equation, one uses the identity $\partial_\phi^2+\partial_\theta^2 = \partial_z\partial_{\bar{z}}$ and

$$\tilde{E}_t = -2\,\mathrm{Im}(z^{-1})\,e^{-z\bar{z}/t}\ .$$

The solution $\tilde{\omega}_t = J^{-1/2}\tilde{E}_t$ thus obtained is not yet 2π-periodic in θ and hence does not lift to a function on the Abelian group $A = \mathbb{R}_+\times\mathrm{U}(1)$. Enforcing periodicity by summing over images, we obtain:

$$\omega_t(\phi,\theta) = J^{-1/2}(\phi,\theta)\sum_{n\in\mathbb{Z}}\frac{4\theta_n}{\phi^2+\theta_n^2}\,e^{-(\phi^2+\theta_n^2)/4t}\ ,$$

where $\theta_n = \theta + 2\pi n$ as before. This is the correct answer, and it is easily seen to coincide exactly with the semi-classical result for the partition function derived above.

As emphasized above, the coincidence of the semi-classical expansion with the exact expression for the partition function is a particular feature of the group manifold structure of the σ-model for the symmetry class CI. As a result, we can deduce the existence of other novel symmetry classes where the transport properties of the quasi one-dimensional system can be inferred from the structure of the semi-classical expansion.

III. GENERALIZATIONS

Having established the principle, we now turn to consider extensions of the present scheme to other symmetry classes. Here we consider classes DIII and AIII:

A. Class DIII (even)

Symmetry DIII is realized[15] in superconductors which exhibit time-reversal symmetry but where the SU(2) spin rotation symmetry is broken by a spin scattering mechanism such as a spin-orbit interaction. The latter presents a critical testing ground for the present theory as it has already been established by Brouwer et al.[6] that exponential localization is *absent* generically in quasi one-dimensional systems of this symmetry class.

For the symmetry class DIII, the corresponding target space of the nonlinear σ-model is a Riemannian symmetric superspace of type $C|D$[13]. This means that we are to work again with the complex orthosymplectic Lie supergroup, $G = \mathrm{OSp}(2n|2n)$, but now one must select a non-compact symmetric space $\mathrm{Sp}(2n,\mathbb{C})/\mathrm{USp}(2n)$ in the boson-boson sector and the compact group $\mathrm{SO}(2n)$ in the fermion-fermion sector. For $n = 1$ these spaces are isomorphic to H^3 (the three-hyperboloid) and U(1) respectively. A maximal commuting subgroup $A = \mathbb{R}_+ \times \mathrm{U}(1)$ is still formed by diagonal matrices $a = \mathrm{diag}(e^{\phi}, e^{-\phi}, e^{i\theta}, e^{-i\theta})$ with $\phi \in \mathbb{R}$ and $\theta \in [-\pi, \pi]$, so the saddle-point configurations (14) are unchanged. Only the fluctuation contribution differs.

The parametrization of the target space by the exponential mapping $\tilde{g} = \exp X$ is the same as for class CI but with the boson-boson and fermion-fermion sectors interchanged:

$$X = \begin{pmatrix} e & b & \delta & \beta \\ c & -e & -\gamma & -\alpha \\ \alpha & \beta & d & 0 \\ \gamma & \delta & 0 & -d \end{pmatrix} ,$$

where d is now imaginary, e real, and $c = \bar{b}$. In view of the duality (by exchange of the compact and non-compact sectors) connecting the nonlinear σ-models for the classes CI and DIII, all calculations for DIII are very similar to those for CI and, for brevity, we simply quote the results here. The partition function, obtained by performing the sum over geodesics together with the integral over Gaussian fluctuations, is given by

$$\mathcal{Z}_T(\phi,\theta) = \sum_{n\in\mathbb{Z}} \frac{\sinh\left(\frac{1}{2}(\phi + i\theta_n)\right)}{\frac{1}{2}(\phi + \mathrm{i}\theta_n)} \frac{\sinh\left(\frac{1}{2}(\phi - i\theta_n)\right)}{\frac{1}{2}(\phi - i\theta_n)} \times \frac{\phi}{\sinh\phi}\, e^{-(\phi^2+\theta_n^2)/4T} . \qquad (20)$$

From Rejaei's relation, the corresponding density of transmission eigenvalues is then given by

$$\rho_T^{D\mathrm{III}}(\phi) = \frac{1}{2T} + \frac{1}{2(\phi^2+\pi^2)} - \sum_{n\neq 0} \frac{e^{-n(n+1)\pi^2/T}}{2\pi^2 n} \,\mathrm{Re}\, \frac{\phi + i\pi}{\phi + i\pi(n+1)}\, e^{in\pi\phi/T} . \qquad (21)$$

A plot of this function in Fig. 1 shows the 'crystallization' of transmission eigenvalues for $t \gg 1$. Yet, exponential localization does not take place, as $\rho_T(\phi)$ peaks at $\phi = 0$ (maximal transmission), and the peak amplitude decays only algebraically with increasing T.

Turning to the mean conductance, the partition function yields the following expression:

$$\begin{aligned} C(T) &= \frac{1}{T} + \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\, e^{-n^2\pi^2/T} \\ &= \frac{2}{\sqrt{\pi T}} + \frac{2}{\sqrt{\pi}} \sum_{l=1}^{\infty} \int_T^{\infty} e^{-l^2\tau} \frac{d\tau}{\tau^{3/2}} . \end{aligned} \qquad (22)$$

The first expression agrees with the result of Ref. 6; the second is obtained from it by Poisson resummation. The first two terms, $C^{\mathrm{pert}}(T) = t^{-1} + 1/3$, represent the Ohmic and weak anti-localization terms that can be computed from a standard perturbation theory for the mean conductance. As with symmetry class CI, higher-order corrections from the perturbative expansion vanish identically. All non-zero corrections are non-perturbative and arise from the non-trivial geodesics in the compact sector. The second expression in Eq. (22) presents an anomalous diffusive asymptotics for large $T = L/\xi$:

$$C(T) \approx \frac{2}{\sqrt{\pi T}} \qquad (T \gg 1) ,$$

a feature already seen in Ref. 6.

Finally, expressed in the form of the heat kernel, the solution for class *D*III is trivially related by the duality discussed earlier and the validity of the expression for the partition function above may be confirmed straightforwardly. As a final application of semi-classical exactness, we turn now to one of the chiral symmetry classes.

B. Class *A*III

This symmetry class is relevant to the low-energy physics of the chiral Dirac operator[27] and to the random flux model[28]. Lately, it has been discussed[29,30] in the context of the quasi-particle properties of a *d*-wave superconductor subject to a smooth random potential. Previous studies of quasi one-dimensional systems in this class[31,32,33,34] have revealed a surprisingly rich behaviour, including localization-delocalization transitions with a number of critical points.

1. Rejaei relation

Before exploring the properties of the nonlinear σ-model action, it is first necessary to confirm the form of Rejaei's relation for this symmetry class. For this we refer to the σ-model formulation of the bond disordered quasi one-dimensional chain given by Altland and Merkt[34]. We consider N chains of $2M \gg N$ sites with hopping matrix element $t_{\mathrm{h}}^{(m)} = 1 + (-)^m a$ along the chains between sites $2m$ and $2m+1$, where a is a 'staggering' parameter. In the following, we suppose that a weak disorder potential (which, further, lifts time-reversal symmetry) of strength $\ll 1$ couples the chains. We view the single particle Hilbert space as the product $\mathsf{H} = \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^M$, with the first factor referring to the odd and even numbered sites along the chain.

The sublattice symmetry of the Hamiltonian translates to the condition $H = -\Sigma_3 H \Sigma_3$, where Σ_i represent the Pauli matrices in this odd/even subspace. From this sublattice symmetry there follows the relation

$$G^A_{-\epsilon} = -\Sigma_3 G^R_\epsilon \Sigma_3 .$$

The corresponding current operator is simply given by $\Sigma_2/2$ (if we ignore the disorder and staggering, or evaluate it in the clean, unstaggered leads). Thus the moments of the transmission matrix at zero energy are given by

$$\begin{aligned} \mathrm{tr}(\mathbf{t}_0\mathbf{t}_0^\dagger)^n &= \mathrm{tr}_{\mathsf{H}}(\hat{v}_L G_0^A \hat{v}_R G_0^R)^n \\ &= \mathrm{tr}_{\mathsf{H}}(\mathcal{P}_1 \Sigma_2 G_0^A \mathcal{P}_M \Sigma_2 G_0^R)^n \\ &= -\mathrm{tr}_{\mathsf{H}}(\mathcal{P}_1 \Sigma_1 G_0^R \mathcal{P}_M \Sigma_1 G_0^R)^n, \end{aligned} \qquad (23)$$

where $\mathcal{P}_m$ projects onto the $2m^{\text{th}}$ site of each chain. It seems that these moments cannot in general be presented as the coefficients in some expansion of a functional determinant. The average conductance ($n = 1$) may however be obtained as the $\phi_L\phi_R$ coefficient of the determinant of a hamiltonian with hopping $e^{i\phi_L}$ between the 1^{st} and 2^{nd} sites, and $e^{i\phi_R}$ between the $2M-1^{\text{th}}$ and $2M^{\text{th}}$ sites (since hopping is described by a Σ_1 term in the Hamiltonian).

We may thus study the usual partition function

$$\mathcal{Z}_T(\phi,\theta) = \int\limits_{g_0=\mathbf{1}}^{g_T=a(\phi,\theta)} \mathcal{D}g \, \exp\left(-\frac{1}{4}\int\limits_0^T \mathcal{L}dt\right) ,$$

for $a(\phi,\theta) = \mathrm{diag}\left(e^{i\phi}, e^{i\theta}\right)$ (the matrix g being 2×2). The conductance can be then identified with the coefficient $C(T)$ in the expansion

$$\begin{aligned} \mathcal{Z}_T(\phi,\theta) = 1 + C_0(T)(\phi - i\theta) - C(T)(\phi^2+\theta^2)/4 \\ + C_1(T)(\phi - i\theta)^2 + \ldots \end{aligned} \qquad (24)$$

2. Semi-classical calculation

The most general nonlinear σ-model Lagrangian for quasi one-dimensional systems belonging to class *A*III (and in suitable units of length) depends on two parameters u, v:

$$\mathcal{L} = \mathrm{STr}\,(g^{-1}\dot{g})^2 + 4v\,\mathrm{STr}\,g^{-1}\dot{g} - u(\mathrm{STr}\,g^{-1}\dot{g})^2 .$$

The target space of the nonlinear σ-model for this class[13] is of type $A|A$ and is obtained from the complex Lie supergroup $\mathrm{GL}(n|n)$ by picking $M_{\mathrm{bb}} = \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ in the boson-boson sector and $M_{\mathrm{ff}} = \mathrm{U}(n)$ in the fermion-fermion sector. For $n = 1$ these are $\mathrm{GL}(1,\mathbb{C})/\mathrm{U}(1) \simeq \mathbb{R}_+$ and $\mathrm{U}(1)$ respectively. The exponential parametrization $\tilde{g} = \exp X$ of the target space is achieved by setting

$$X = \begin{pmatrix} p & \alpha \\ \beta & iq \end{pmatrix}$$

with real commuting variables p, q and anti-commuting variables α, β. Again, $a(\phi,\theta)$ parameterizes the maximal Abelian subgroup. Based on the geodesics $a_t^{(n)} = \mathrm{diag}\left(e^{\phi t/T}, e^{i\theta_n t/T}\right)$ with $\theta_n = \theta + 2\pi n$, the semiclassical approximation for $\mathcal{Z}_T(\phi,\theta)$ is found to be

$$\mathcal{Z}_T(\phi,\theta) = \sum_{n\in\mathbb{Z}} (-1)^n \frac{\sinh\left(\frac{1}{2}(\phi - i\theta)\right)}{\frac{1}{2}(\phi - i\theta_n)} \, e^{-v(\phi-i\theta_n)-[\phi^2+\theta_n^2-u(\phi-i\theta_n)^2]/4T} \; . \tag{25}$$

Expanding the result above for the partition function, and using Eq. (24) gives

$$\begin{aligned} C(T) &= \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty}(-1)^n \cos(2\pi n v)\, e^{-(1+u)n^2\pi^2/T} \\ &= \frac{1}{\sqrt{\pi T(1+u)}} \sum_{l\in\mathbb{Z}} e^{-(l+v-1/2)^2 T/(1+u)} \; , \end{aligned} \tag{26}$$

where the second expression is obtained by Poisson resummation of the first.

The result (26) depends periodically on the parameter $\theta \equiv 2\pi v$ with period 2π. In fact, θ has the meaning of a topological angle. From Ref. 34 we know that tuning θ from 0 to 2π amounts to changing by two the number of channels (which is assumed to be large in order for the σ-model approximation be valid). In terms of the bond disordered chains discussed earlier

$$\theta = \pi\,(N - f) \mod 2\pi,$$

where f is the staggering parameter a scaled by the strength of the disorder. We see that without staggering, a delocalized state appears only for odd channel number. The localization length diverges with critical exponent 1 as the staggering approaches the critical values $f = N - 1 \pmod 2$.

If class AIII is realized by Dirac fermions in a random Abelian gauge field, the parameter $u > 0$ measures the strength of the gauge disorder. Although the Lagrangian $\mathcal{L}$ at first sight would seem to become unstable at $u = 1$ (the coefficient of $\dot{p}^2$ becomes negative there), this is not really so. What matters is the *full* quadratic form

$$\mathcal{L}_{v=0} = (1-u)\dot{p}^2 + (1+u)\dot{q}^2 + 2iu\dot{p}\dot{q} + \text{odd variables}\;,$$

which has the Jordan normal matrix form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, and is invertible for any u. In fact, the final result (26) reveals a smooth dependence of the mean conductance on u for all physical values ($u > 0$) including the fake singularity $u = 1$.

3. *Heat Kernel*

Finally, seeking an exact solution through the heat kernel, an exact calculation of $\omega_T(\phi,\theta)$ is still possible as the Hamiltonian $\mathcal{H}$ obtained by canonical quantization from $\mathcal{L}$ can once again be transformed to Euclidean form:

$$\begin{aligned} \mathcal{H} = -J^{-1/2} \big(&\partial_\phi\partial_\phi + \partial_\theta\partial_\theta + 2v(\partial_\phi - i\partial_\theta) \\ &+ u(\partial_\phi - i\partial_\theta)^2\big)\, J^{1/2} \;, \end{aligned}$$

with $J = 1/\sinh^2\left(\frac{1}{2}(\phi - i\theta)\right)$. By solving the differential equation $\partial_t\omega_t = -\mathcal{H}\omega_t$ with the appropriate δ-function initial conditions at $t = 0+$, we exactly recover the semiclassical answer (25).

IV. CONCLUSION

We have confirmed the exactness of our semiclassical analysis for symmetry classes CI, DIII, and AIII. By simply summing over the saddle points of the classical action — recall that these differed by the length of the geodesic looping around the compact fermion-fermion submanifold of the theory — and treating the fluctuations in the Gaussian approximation, we have obtained the exact result for the partition function.

The underlying reason for the exactness of our calculations is that the path integral on the σ-model manifolds considered here satisfies an infinite-dimensional generalisation of the Duistermaat-Heckman theorem[23,24]. That is, the integration manifold in Eq. 13, being related to the space ΩG of based loops in G, is symplectic with Liouville measure, and the action is the momentum mapping on this manifold.

We have discussed all the symmetry classes where the σ-model target space has a group-like structure, and thus exhausted the situations in which the semiclassical approach is exact. Nevertheless, we hope that this approach may inform future investigations of more taxing localization problems. One very interesting extension that could be tackled in a similar manner is the critical scaling of the localization length near the band center in situations where zero energy is described by class DIII or AIII. This lies outside of the scope of methods like the DMPK equation.

APPENDIX: EMBEDDING OF g IN Q FOR CLASS CI

Let us review the structure of the σ-model target space for Class CI. The construction used in Ref. 13 is based on 'copying' the symmetries of the Hamiltonian to symmetries of the auxiliary space of the Q-field. For Class CI (particle-hole and time reversal symmetry), this is achieved by demanding that the fields satisfy

$$\begin{aligned} \Psi_M &= \mathcal{C}\bar{\Psi}_M^{\mathrm{T}}\gamma^{-1}, \quad \bar{\Psi}_M = -\gamma\Psi_M^{\mathrm{T}}\mathcal{C}^{-1}, \\ \Psi_M &= \bar{\Psi}_M^{\mathrm{T}}\tau^{-1}, \quad \bar{\Psi}_M = \tau\Psi^{\mathrm{T}}. \end{aligned} \tag{A.1}$$

The subscript M is to remind us that in the formulation of Ref. 13 the variables Ψ_M and $\bar{\Psi}_M$ are (super)*matrices* mapping the auxiliary space to the physical Hilbert space (where the Hamiltonian acts), and vice versa. In the present case we work with the more traditional column and row supervectors Ψ_V and $\bar{\Psi}_V$, since:

- In the localization problem the $Q(\mathbf{r})$ field used to decouple the term arising from potential disorder retains the Hilbert space structure of the Hamiltonian (particle hole space and spatial index), whereas in the random matrix problem that structure is lost on averaging and, more importantly,
- We use some of the additional structure (the cc space) to enlarge our Hamiltonian (Eq. 7).

Transcribing the conditions (A.1) for supervectors gives

$$\begin{aligned} \Psi_V &= \gamma \mathcal{C} \bar{\Psi}_V^{\mathrm{T}}, \quad \bar{\Psi}_V = -\Psi_V^{\mathrm{T}} \mathcal{C}^{-1} \gamma^{-1}, \\ \Psi_V &= \tau \bar{\Psi}_V^{\mathrm{T}}, \quad \bar{\Psi}_V = \Psi^{\mathrm{T}} \tau^{-1}, \end{aligned} \tag{A.2}$$

where we used the fact that γ and τ can be chosen to be orthogonal matrices. Now in the localization problem $Q(\mathbf{r})$ is a 16×16 matrix (ph×cc×tr×bf - by 'tr' we mean the space to accommodate time reversal symmetry), so we need to check that the conditions (A.2) give rise to a σ-model with the same 8×8 space as in the random matrix problem. This is straightforwardly verified following Ref. 22 by observing that $Q(\mathbf{r}) \sim \Psi \otimes \bar{\Psi}\sigma_3$ (we drop the V subscript from now on), arising from a decoupling of potential disorder, has the symmetry

$$\begin{aligned} Q &= \sigma_1 \gamma Q^T \gamma^{-1} \sigma_1 \ , \\ Q &= \sigma_3 \tau Q^T \tau^{-1} \sigma_3 \ . \end{aligned} \tag{A.3}$$

The saddle-point manifold is spanned by

$$Q = W \sigma_3 \Sigma_3 W^{-1}, \qquad W = w \otimes \mathbb{1}_{\mathrm{ph}},$$

or

$$Q = \sigma_3 q, \qquad q = w \Sigma_3 w^{-1}.$$

Thus (A.3) implies that the symmetries of the 8×8 field q are

$$\begin{aligned} q &= -\gamma q^T \gamma^{-1}, \\ q &= \tau q^T \tau^{-1}, \end{aligned} \tag{A.4}$$

which are the same relations found in Ref. 13. To construct an embedding of the 4×4 matrix group-valued field g in q requires an explicit choice of γ and τ. We use

$$\begin{aligned} \gamma &= E_{\mathrm{bb}} \otimes \gamma_{\mathrm{b}} + E_{\mathrm{ff}} \otimes \gamma_{\mathrm{F}}, \\ \tau &= E_{\mathrm{bb}} \otimes \tau_{\mathrm{b}} + E_{\mathrm{ff}} \otimes \tau_{\mathrm{f}}, \end{aligned}$$

$$\begin{aligned} \gamma_{\mathrm{b}} &= \Sigma_1 \otimes \tau_3, \quad \gamma_{\mathrm{f}} = i\Sigma_2 \otimes \mathbb{1}_{\mathrm{tr}}, \\ \tau_{\mathrm{b}} &= \mathbb{1}_{\mathrm{cc}} \otimes \tau_1, \quad \tau_{\mathrm{f}} = \Sigma_3 \otimes i\tau_2, \end{aligned}$$

where τ_i are the Pauli matrices in tr space, and E_{bb} and E_{ff} are projectors in the bose-bose and fermi-fermi sectors. Note that the structure of the vector in cc space in the bosonic sector is in fact then *not* given by Eq. 6, but rather looks like

$$\Psi_B = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_B \\ \mathcal{C}\tau_3 \bar{\psi}_B^T \end{pmatrix}_{\mathrm{cc}}, \qquad \bar{\Psi}_B = \frac{1}{\sqrt{2}} \left(\bar{\psi}_B, -\psi_B^T \tau_3 \mathcal{C}^{-1} \right)_{\mathrm{cc}}.$$

Fortunately this doesn't change the boundary conditions. Now the correct embedding is obtained by finding a transformation to diagonalize the matrix

$$\eta \equiv -i\gamma\tau^{-1} = \Sigma_1 \otimes \tau_2 \otimes \mathbb{1}_{\mathrm{bf}},$$

$$\eta \longrightarrow U^\dagger \eta U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \otimes \mathbb{1}_{\mathrm{bf}},$$

while simultaneously sending

$$\Sigma_3 \otimes \mathbb{1}_{\mathrm{tr}} \otimes \mathbb{1}_{\mathrm{bf}} \longrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_{\mathrm{bf}}.$$

Why? The point is that the set of w's that preserve the symmetries (A.4) takes the form

$$w \longrightarrow \mathrm{diag}(g_1, g_2), \qquad g_1, g_2 \in \mathrm{OSp}(2|2), \tag{A.5}$$

in such a basis. The stability group of transformations that send $w\Sigma_3 w^{-1} \to \Sigma_3$ is then given by Eq. A.5 with $g_1 = g_2$, so that the saddle-point manifold is parameterized by

$$q = U \mathrm{diag}(g, g^{-1}) U^\dagger \Sigma_3, \qquad g \in \mathrm{OSp}(2|2) \ .$$

An explicit form for U is

$$U = \frac{1}{2} \begin{pmatrix} -i & -1 & i & 1 \\ 1 & i & -1 & -i \\ -i & 1 & -i & 1 \\ 1 & -i & 1 & -i \end{pmatrix} \otimes \mathbb{1}_{\mathrm{bf}}.$$

Let us verify that this takes the maximal abelian subgroup A to the q matrices of the form (9):

$$\begin{aligned} q &= U \mathrm{diag}(a(\phi,\theta), a^{-1}(\phi,\theta)) U^\dagger \Sigma_3 \\ &= \mathrm{diag}\left(\begin{pmatrix} \cosh\phi & -\sinh\phi \\ \sinh\phi & -\cosh\phi \end{pmatrix}, \begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} \right)_{\mathrm{bf}} \otimes \mathbb{1}_{\mathrm{tr}}. \end{aligned}$$

Finally, the form of the action (13) in terms of g follows from

$$\begin{aligned} \mathrm{STr}\left[\nabla q\right]^2 &= 2\,\mathrm{STr}\left[\nabla g \nabla g^{-1}\right] \\ &= -2\,\mathrm{STr}\left[g^{-1}\nabla g\right]^2. \end{aligned}$$

* alamacra@princeton.edu

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